On discrimination in auctions with endogenous entry*

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Abstract

When entry is exogenous, strong buyers should be discriminated against weak buyers to maximize revenues (Myerson 1981). When entry is endogenous so that entrants' expected payoffs do not depend on the proposed mechanism, optimal discrimination takes a completely different form. The revenue-maximizing equilibrium requires that there should be no discrimination with respect to entrants irrespective of their ex-ante characteristics. Besides, those buyers who always participate should be discriminated against entrants independently of their strength. These predictions are independent of the equilibrium selection when the number of potential entrants grows large. The optimality of first-price auctions is also discussed.

Keywords: auctions with endogenous entry, asymmetric buyers, discrimination, procurements, bid preference programs, optimal auction design, incumbents, favoritism.

JEL classification: D44, H57, L10.

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1 Introduction

In procurement auctions, governments are always tempted to use discrimination in favor of some bidders, for example according to whether a firm is domestic, whether a firm has already got a contract in the recent past, or according to the size of the firm or also according to some affirmative action criteria (in particular in favor of minority-owned businesses).[1] A legitimate fear is that discrimination would distort competition in an inefficient way, and that it would lead to pay a higher price for the same service. This is an argument often put forward by the World Trade Organization or the European Commission to ban discrimination.[2] More generally, questions of discrimination are an essential part of the policy debate surrounding procurement auctions, and it is of practical major importance to understand the pros and cons of discrimination (see Mattoo (1996) for an account of how discrimination issues play a central role in policy debates).

From a theoretical viewpoint, if firms are ex ante asymmetric, it would seem some discrimination could be desirable to the extent that it would induce a more balanced competition. The work of Myerson (1981) on optimal auctions can be interpreted as providing some support to this idea by giving a precise measure of how stronger bidders should be handicapped to generate more revenues (see McAfee and McMillan (1989) for proposing such an interpretation of Myerson's work and Corns and Schotter (1999) for a corresponding experimental investigation meant to support affirmative action plans).

The work of Myerson (1981) assumes that the set of bidders in the auction is exogenously given. But, if entering the procurement is too costly to attract every possible firm, participation should be viewed as being endogenously determined. This adds another consideration. When entry is endogenous, how many and which buyers/firms show up typically depends on the (procurement) auction format, and thus on the form and magnitude of discrimination. It is then important to reassess the extent to which discrimination is desirable in the presence of asymmetries when entry is endogenous.

Mixing considerations of endogenous entry and possible asymmetries in auctions is a topic of great practical importance. For example, in the context of the European 3G telecom auctions that took place in the early 2000, Paul Klemperer has suggested that in the presence of asymmetric bidders the use of second-price (or ascending price) auctions could have undesirable consequences on revenues to the extent that it would make weaker bidders too unlikely to win the auction and thus lead such bidders not to participate. This has led him to recommend the use of Anglo-Dutch auctions where an ascending auction would be followed by a sealed bid first-price auction

[1] Mougeot and Naegelen (1989) report that the Buy American Act (which starts in 1933) promotes bid subsidies ranging from 6 percent to 12 percent. Defense contracts have a special treatment and subsidies can be as high as 50%. Canadian and Australian legislations have similar dispositions. In other countries, e.g. European countries, favoritism with respect to domestic firms is not written in the law, but non-explicit discrimination rules lead to the same results.

[2] The WTO which struggles against barriers to trade, rules out discrimination in its Agreement on Government Procurement, the Buy American Act being a notable exception. The European Commission cares about helping SMEs winning public procurements but only through non-discriminatory approaches (see http://ec.europa.eu/enterprise/policies/sme/small-business-act/).

among the two bidders left from the first stage: The hope was that such a format would preserve most of the efficiency advantages of the second-price auction while making participation more attractive to weaker bidders (by giving such bidders more chance to win the auction). One output of our analysis will be to shed light on whether such formats (which typically induce a form of indirect discrimination by helping weaker bidders) turn out to be desirable from a revenue viewpoint. More generally, a number of scholars including Klemperer (2002) and Milgrom (2004) have stressed the practical importance of understanding the consequences of endogenous entry in contexts with asymmetries.[3] Yet, general models of auctions with endogenous entry in the presence of asymmetries between bidders have not been systematically analyzed so far presumably because such models were considered to be untractable.[4] Our analysis will fill this gap, which is essential for the understanding of discrimination.

From both an empirical and an econometric perspective, the interaction of auction design with endogenous entry has received considerable attention in the recent past.[5] For example, Marion (2007) establishes that in California the five percent subsidy that accrues to small businesses in auctions for road construction projects using only state funds increases the procurement costs by 3.8 percent compared to projects using federal aid where there is no such bid preference program. Marion (2007) observes that the main channel for this detrimental effect of discrimination comes from the reduced participation of large firms with low cost in those auctions in which they are handicapped. In another context, Athey et al. (2013) find a positive effect of discrimination on revenues. Specifically, according to their structural estimates for timber auctions, the seller's revenue is increasing in the subsidy level on small firms, at least up to a 20% subsidy. Our study will shed light on these empirical findings by establishing that if the entry decisions of some firms are inelastic to the mechanism, these should be discriminated against, but if the rents of other firms are fixed irrespective of the mechanism (through the working of endogenous entry), such firms should not be discriminated against no matter how strong they are.

The environment we consider can be described as follows. A seller of a good commits to a mechanism or auction format before the various potential buyers make their decision whether or not to enter. Buyers may be potential entrants coming from various groups with possibly different distributions of valuations and different entry costs where we assume each group is composed of several potential entrants (this number will be assumed to be not too small so that it would not be an equilibrium for all potential entrants to enter with probability 1). Alternatively, buyers may be isolated buyers in which case we refer to them as incumbents. When a buyer chooses to enter, this

[3]"Many of the most important practical issues in auction design concern the interaction of the design and entry decisions. [...] Models with entry and asymmetric bidders have received much less attention than symmetric models, despite the great influence of asymmetries among bidders on entry.", Paul Milgrom (2004). "Much of what we have said about auction design is no more than an application of standard antitrust theory. The key issues in both fields are collusion and entry.", Paul Klemperer (2002).

[4]One would typically expect to deal with the entry decisions of several groups of buyers for all possible mechanisms, thereby leading to a problem of too great complexity.

[5]See e.g. Athey et al. (2011,2013), Gentry and Li (2014), Krasnokutskaya and Seim (2011), Li (2005), Li and Zheng (2009), Marion (2007), Marmer et al. (2013), Roberts and Sweeting (2012,2013).

buyer is assumed to learn his valuation immediately after the entry decision, thereby ensuring that we are in a private value setting at the bidding stage. The seller may observe signals regarding the profile of entrants, thereby allowing her to use explicit discriminatory devices (on the top of indirect discriminatory devices). We also allow for information structures in which participants' private signals are correlated.

Our first result says that if the seller seeks to maximize revenues and there are no incumbents, one equilibrium (in fact the one most preferred by the seller) is such that the seller posts a second-price auction with a reserve price set at her valuation and that the entry probabilities of the various potential entrants are chosen to maximize the expected total welfare net of the entry costs. We note that the same result holds true if there are incumbents but the objective of the seller includes the profits of the incumbents in addition to the revenues. Given that such a second-price auction induces no discrimination (either directly or indirectly), we conclude that discrimination is not desirable from a revenue viewpoint, at least when there are no incumbents and the entrants follow the equilibrium most preferred by the seller.

Our second result considers a revenue-maximizing seller in a situation with entrants but also with incumbents. We also assume for that part that the valuation of any incumbent is distributed independently of any other information held by other buyers as in Myerson (1981) while imposing no restriction about the information that buyers have about the entrants in the auction. This part mixes the theory of Myerson (1981) in which it is known that some form of discrimination is optimal with the situation discussed above with entrants only for which our first result establishes that discrimination is not desirable. While one might have thought that mixing the two problems could lead to choose some discrimination in favor of some groups of entrants in an attempt to reduce the rents left to the incumbents, it turns out that the optimal mechanism (under an extra assumption w.r.t. incumbents' entry costs) takes a different and in fact very simple form: Use the distortion Myerson (1981) introduced for incumbents (replacing incumbents' valuations by their virtual valuations) and use no discrimination for entrants (no matter which group they come from). More precisely, one equilibrium (in fact the one most preferred by the seller) is such that: 1) The seller uses a generalized second-price auction, referred to as the virtual pivotal mechanism, where the allocation rule is such that the good is assigned to the bidder with the highest valuation (which includes the seller in which case it corresponds to her reservation value) with the twist that for incumbents -and only for them- valuations are replaced by virtual valuations à la Myerson (1981); 2) The entry probabilities are chosen to maximize the expected virtual welfare (in which the valuations of the incumbents are replaced by their virtual valuations) net of the entry costs. Such a result gives thus a clear cut answer to the form of discrimination to be adopted by a revenue maximizing seller. She should not discriminate at all against or in favor of those buyers whose entry rates adjust so that their utilities remain constant (as is the case for entrants), and she should discriminate against buyers whose entry rates are independent of the auction format (at least locally) as is the case here for incumbents.

While the above results are suggestive of what discrimination should optimally look like, they leave the door open to the possibility that there could be other equilibria in which discrimination could take a different form. In the final part of the paper, we consider the case in which the number of potential entrants in each group is large (in fact we work directly in the limit where the number of entrants in each group is unbounded). In such a case, we make the important observation that in the absence of incumbents the above equilibrium outcome without discrimination (that is the one most preferred by the seller) is in fact the only possible equilibrium outcome. In the presence of incumbents with sufficiently low entry costs, the above equilibrium outcome as resulting from the virtual pivotal mechanism is also the unique equilibrium outcome. Such a uniqueness result has no counterpart in the literature. It establishes in a strong way for the case of large pools of potential entrants and no incumbents that discrimination whatever its form would be harmful for revenues.

While our setup is extremely general, there is a simple intuition explaining our results. First, the equilibrium utility of entrants is bounded from below by their entry costs (as otherwise they would not participate). This implies that, when there are no incumbents, the objective of a revenue-maximizing seller is bounded from above by the welfare net of the entry costs. As a result, the objective of the seller can never outperform the unconstrained maximum of the net welfare obtained from the maximization with respect to the choice of allocation rule and the choice of entry rates. We refer to such an upperbound as the first-best net welfare. Second, the second-price auction with a reserve price set at the seller's valuation, referred in the sequel to as the efficient second-price auction, has the property of giving the right incentives regarding the entry decision in the sense that the entry game can be a viewed as a potential game in which the potential function is the net welfare. This in turn implies that the welfare maximizing entry rates are an equilibrium profile when the chosen mechanism is the efficient second-price auction. Because the potential function of the entry game coincides with the objective of the seller and the efficient second-price auction also allocates the good efficiently for a given profile of participation, it follows that one equilibrium is for the seller to use the efficient second-price auction and that entry probabilities are chosen to maximize the net welfare (essentially, because, as explained above, there is no hope the seller could achieve in any other mechanism an expected revenue that is strictly larger that the first-best net welfare).

Such an intuition explains the results previously obtained by Levin and Smith (1994), McAfee (1993), Peters (1997, 2001) or Peters and Severinov (1997) in relation to the equilibrium choice of reserve prices in second-price auctions either in contexts with endogenous entry (as we consider it) or in contexts with competing auctions. These authors all observe in seemly different contexts (Levin and Smith (1994) and McAfee (1993) consider models without ex ante information or with full ex ante information on valuations, respectively) but with ex ante symmetric entrants from sellers' perspective, that sellers would find it optimal to set reserve prices at their valuation in second-price auctions in contrast to Myerson (1981) who showed that optimal reserve prices should be set strictly above the seller's valuation in contexts with fixed sets of participants. By allowing for ex ante asymmetries between entrants and also by allowing for any kind of information structure on what is observed by the seller, our analysis permits to shed light on the issue of discrimination that the previous literature limited to symmetric entrants could not address.

Allowing for the presence of incumbents (and still assuming the objective is revenue) would seem to invalidate the above arguments. But, using transformations similar to the ones introduced by Myerson (1981) leads in turn to the same type of arguments after noting that the objective of the seller can be re-written as the net virtual welfare (as described above).

Moving to the case of large pools of potential entrants, we note that the net welfare function in the efficient second-price auction is a globally concave function of the entry rates of potential entrants, thereby ensuring that there is a unique equilibrium outcome of the entry game when the seller uses the efficient second-price auction and there are no incumbents. This in turn yields the announced insight that no discrimination (either direct or indirect) can be desirable in this case whatever the equilibrium when there are no incumbents. It is a priori surprising that one would avoid the multiplicity issue when entrants may come from different groups (the entry game can be thought of as a coordination game in which multiplicity is typically expected). Yet, in the case of large pools, an entrant whatever the group he comes from expects to be facing the same distribution of competitors (which is not so in the case of small pools of entrants), which in turn eliminates the coordination multiplicity that could arise otherwise.

Finally, in the same contexts of large pools of entrants and with extra assumptions on the information structure (that are standard in the auction literature), we note that the seller could use first-price auctions instead of second-price auctions (still with the reserve price set at her valuation). This observation has some applied appeal given that most procurement auctions operate according to the rule of the sealed-bid first-price auctions (even if sometimes biased in favor of some classes of bidders through the use of bid subsidies) and one might have thought such formats would involve some allocative inefficiencies in the presence of ex ante asymmetries between potential entrants. The reason for the optimality of unbiased first-price auctions is that despite the fact that entrants may come from different asymmetric groups, when the set of potential entrants is large, each auction participant expects to be facing the same distribution of competing bids, and therefore his optimal bid is independent of the group he comes from (it depends solely on his valuation). This in turn implies that the first-price auction and the second-price auction lead to the same final assignment in equilibrium and thus also to the same outcome in terms of expected payoff (by revenue equivalence thanks to the extra assumption of non-correlation of private information we make in this part), which also implies that the participation incentives are the same in both formats, hence the conclusion.

The rest of the paper is organized as follows: Section 2 introduces the endogenous entry model with a finite set of asymmetric potential entrants and the equilibrium concept we use where ex ante symmetric bidders are assumed to behave in the same way. The existence of an efficient equilibrium without discrimination is established in Section 3 when there are no incumbents or when the rents of the incumbents are fully internalized in the seller's objective. In Section 4, we characterize the optimal auction in the presence of incumbents when the information structure with respect to incumbents' valuations is the same as in Myerson (1981). To obtain further results while still keeping a large degree of generality, we move in Section 5 to the Poisson model which corresponds to the limit environment where the number of potential entrants in each group goes to infinity. We briefly touch on how our results extend to environments in which the seller is privately informed of her valuation in Section 6.1 and to environments with multiple objects for sale in Section 6.2. Section 7 concludes. Most of the proofs are relegated to the Appendix.

2 The model

2.1 The economic environment

Timing of the game

One seller has one object for sale. Her valuation -denoted by X- is assumed to be known to everybody. We will later on extend our results to the case in which the valuation of the seller is privately known to her. The seller announces an auction mechanism m to which she is committed. This mechanism m must belong to some set $\mathcal{M}$, and we will elaborate on what the set $\mathcal{M}$ can be later on. Buyers then decide simultaneously whether or not to incur an entry cost (which corresponds equivalently to the expected utility of the given buyer if he chooses an outside option and which could also include physical costs). Upon entering, buyers learn their valuation and both the seller and buyers may receive extra signals related to the valuations of others. Based on their information, buyers decide simultaneously whether or not to participate in the mechanism. If they do not participate, they get 0 utility (their overall utility is negative due to the loss of the entry cost, which is sunk). If they participate, they play the mechanism m initially posted where the seller can also possibly be active. The seller and the buyers are assumed to play according to a Nash-Bayes equilibrium.

Comments: Observe that in our formulation we do not allow the seller to approach the buyers before they have made their entry decision.[6] There are several rationales for such an assumption. First, it is illegal to approach buyers before the official start of the selling procedure in the context of many procurement auctions. Second, the seller may have no initial idea as to which buyer may a priori be interested in the good for sale, so that even if the seller could secretly approach some buyers ex ante, she would have no idea whom to approach, thereby making such ex ante contacts impossible (or too costly). At some point, with the latter rationale in mind, we will discuss the case in which the seller can approach ex ante the incumbents assumed to be well identified but not the entrants assumed to be anonymous and potentially numerous (see the distinction between incumbents and entrants below).

[6] We do not either allow the seller to approach the buyers right after the entry decision but before they have got extra information about their valuation because we have in mind that the extra information comes at the same time as the entry decision. Note however that we allow the buyers to have private information ex ante before they make their entry decisions.

Information and preferences

We assume that there are two classes of buyers. The first class of buyers referred to as "incumbents" are individual buyers. There are $I \geq 0$ of them and incumbents are denoted by $i = 1, \dots I$. Incumbent i is characterized by an entry cost $C_i^I \geq 0$ and a cumulative distribution $F_i^I(.|z)$, from which i's valuation is drawn conditional on the realization z of some underlying variable Z whose distribution is a parameter of the economy. We denote the set of incumbents $\{1, \dots I\}$ by $\mathcal{I}$.

The second class of buyers referred to as potential entrants are divided into $K \geq 0$ different groups, each group k being composed of $\mathcal{N}_k \geq 2$ (ex ante identical) buyers. Each buyer from group $k = 1, \dots K$ is characterized by the cost of entry $C_k > 0$ and the cumulative distribution $F_k(.|z)$ from which his valuation is drawn conditional on the realization of z. We typically think of $\mathcal{N}_k$ as being large enough so that it would not be an equilibrium for all $\mathcal{N}_k$ to enter with probability 1. We denote the set of groups $\{1, \dots, K\}$ by $\mathcal{K}$.

We stress that the only element of private information of a potential entrant before entry is his knowledge about his group. However, we allow for the possibility that some signal $s_Z \in \mathcal{S}_Z$ about z is observed by all buyers before they make their entry decisions.[7] To simplify the presentation and alleviate the notation, we will nonetheless omit this possibility in most of the paper. Conditionally on z, the valuations of the various buyers (both the incumbents and the potential entrants) are assumed to be drawn independently.[8] We assume that there exists $\overline{x} \geq 0$ such that the supports of the distributions $F_i^I(.|z)$ and $F_k(.|z)$ are subsets of $[0, \overline{x}]$. We do not exclude that the buyers who incur the entry cost and/or the seller could observe after entry some signals that are correlated with the realization of the valuations of the other participants and also with the signals observed by the other agents including the seller. In particular, our setup covers situations ranging from the case in which the seller would observe nothing about the pre-entry types of the participants to the case in which she could identify those types perfectly but also to the case in which the seller would observe perfectly the valuations of the participants so that there would be not informational asymmetry after the entry stage. The vector of post-entry signals is denoted by s. We also postulate that based on the profile of participants, some public information s_P on which the mechanism can be made contingent is disclosed.

To sum up, the primitives of the economic environment can be described by $(F_i^I(.|z), C_i^I)_{i \in \mathcal{I}}$, $(\mathcal{N}_k, F_k(.|z), C_k)_{k \in \mathcal{K}}$, the distribution of z, the distribution of the pre-entry public signal s_Z and finally the distribution of the post-entry signals s_P and s conditional on the set of buyers who chose to enter. These are assumed to be common knowledge among the buyers and the seller who are also assumed to be risk neutral.

[7] We could consider alternatively the more general setup in which all the potential entrants from a given group k receive a signal $s_{k,Z}$ about the realization of Z. What is needed though is that there is no heterogeneity between potential entrants inside a group so that all potential entrants from a given group are entirely symmetric at the entry stage.

[8] Conditional independence is a general simple way to introduce some correlation between buyers' valuations.

2.2 The mechanism design setup

The seller chooses a mechanism m in the set of possible mechanisms $\mathcal{M}$: it specifies for each s_P a game form that defines an assignment rule and monetary transfers for each profile of strategies of the bidders (the set of strategy profiles in m is denoted by $\Sigma(m)$). We assume that, in each such mechanism, buyers from the same group of potential entrants are treated in the same way which reflects an anonymity constraint. Besides, to formalize the option not to participate, we assume that every buyer has the option to use a strategy in $\Sigma(m)$ that guarantees him zero payoff. The set of all mechanisms with the above constraints is denoted by $\mathcal{M}^*$. We also assume in the sequel that buyers from the same group follow the same strategy (both in terms of entry decisions and at the bidding stage), which (together with our restriction on mechanisms) implies that all potential entrants from the same group derive the same expected utility from entry. To accommodate situations in which there would be additional constraints on the set of possible mechanisms, we allow for the possibility that the set $\mathcal{M}$ from which the seller can choose her mechanism is a strict subset of $\mathcal{M}^*$.

One can think of discrimination in two ways. Clearly, direct discrimination in which the mechanism would explicitly depend on some observable characteristics of the participants would require that some observation be made by the seller.[9] Another (stronger) view of non-discrimination would require that the final allocation and payments depend on the profile of valuations in an anonymous way. Observe that one may have discrimination in the latter sense (this is sometimes referred to as indirect discrimination) even if the seller receives no signal after entry (and thus cannot rely on direct discrimination). For the sake of illustration, in the context of a first-price auction with exogenous entry among a small set of asymmetric bidders, the outcome of the Nash-Bayes equilibrium induces discrimination in this sense, and the seller need not observe anything about the characteristics of the buyers to be able to implement the first-price auction.[10]

The non-discrimination result that we will obtain considers both aspects. Under some conditions to be expressed below, the seller will not find it optimal to use direct discrimination even if she could. That is, even if the public post-entry information is s_P (including the case of perfect observation of participants' valuations) she would not find it optimal to make use of s_P in her choice of mechanism. Similarly, the seller will not find it optimal either to use indirect discrimination that exploits the underlying asymmetries between buyers or their post-entry signals.

Assuming that the seller can pick any mechanism in $\mathcal{M}^*$ (i.e. $\mathcal{M} = \mathcal{M}^*$) follows the tradition of the mechanism design literature in the vein of Myerson (1981). Let $\mathcal{M}^r_{SP}$ denote the set of second-price auctions, where each auction is characterized solely by the reserve price. Observe that if the seller picks a mechanism in $\mathcal{M}^r_{SP}$ when unrestricted, it means that she chooses not to discriminate among buyers (since all buyers are ex post treated alike in a mechanism $m \in \mathcal{M}^r_{SP}$ provided buyers employ their weakly dominant strategy, which we will assume). Let m^{ESP}_X denote

[9] This is for example the kind of discrimination arising in the optimal auction of Myerson (1981) when the seller observes the identity of buyers whose valuations are drawn from different distributions.

[10] This is also the case of the Anglo-Dutch auction proposed by Klemperer (2002).

the efficient second-price auction in which the seller sets a reserve price r at her valuation X.

2.3 Equilibrium with endogenous entry

A key aspect of our model is the assumption that in each group k of entrants, the (simultaneous) participation decisions are made symmetrically among the (ex ante symmetric) $\mathcal{N}_k$ buyers of the same group and are thus characterized by a single scalar $q_k \in [0,1]$ which corresponds to the probability of entry of a potential entrant from group k. The effective number of entrants from a given group $k \in \mathcal{K}$ of potential entrants is thus taken to be the realization of a random variable following a binomial distribution with $\mathcal{N}_k$ (independent) trials and the probability of success q_k (for each trial). That is, for any $k \in \mathcal{K}$ the probability that there are n_k entrants from group k for all $k \in \mathcal{K}$ is equal to $\prod_{k=1}^{K} \binom{\mathcal{N}_k}{n_k} [q_k]^{n_k} \cdot [1-q_k]^{\mathcal{N}_k - n_k}$.

Before we present the formal definition of equilibrium, some additional notation is required. We let

- $N = (n_1, \ldots, n_K) \in \mathbb{N}^K$ denote a realization of the profile of potential entrants who decide to enter the mechanism. For a given vector N, we let $N_{-k} = (n_1, \ldots, n_{k-1}, n_k - 1, n_{k+1} \ldots, n_K)$ and $N_{+k} = (n_1, \ldots, n_{k-1}, n_k + 1, n_{k+1} \ldots, n_K)$.
- $S \subseteq \mathcal{I}$ denote a realization of the subset of incumbents who decide to enter the mechanism, and $S_{-i} = S \setminus \{i\}$.
- $F^{(j:N \cup S)}$ denote the CDF of the j^{th} order statistic of valuations among the set of entrants $N \in \mathbb{N}^K$ and the subset $S \subseteq \mathcal{I}$ of incumbents choosing to enter.
- $\sigma(m) \in \Sigma(m)$ denote the bidding strategy profile used by all kinds of bidders (both the participating buyers and the seller) in the mechanism m and which depends on their post-entry signals.
- $q = (q_1, \ldots, q_K) \in [0,1]^K$ [resp. $q^I = (q_1^I, \ldots, q_I^I) \in [0,1]^I$] denote the profile of entry probabilities of potential entrants [resp. incumbents].
- $P(N|q) = \prod_{k=1}^{K} \binom{\mathcal{N}_k}{n_k} [q_k]^{n_k} \cdot [1-q_k]^{\mathcal{N}_k - n_k}$ [resp. $P_k(N|q) = \binom{\mathcal{N}_k - 1}{n_k} [q_k]^{n_k} \cdot [1-q_k]^{\mathcal{N}_k - 1 - n_k} \cdot \Big[\prod_{\substack{k'=1 \\ k' \neq k}}^{K} \binom{\mathcal{N}_k}{n_k} [q_k]^{n_k} \cdot [1-q_k]^{\mathcal{N}_k - n_k}\Big]$] denote the probability of the realization $N \in \prod_{k=1}^{K} [0, \mathcal{N}_k]$ for the set of entrants [resp. $N \in \prod_{k'=1}^{k-1} [0, \mathcal{N}_{k'}] \times [0, \mathcal{N}_k - 1] \times \prod_{k'=k+1}^{K} [0, \mathcal{N}_{k'}]$ for the set of entrants among potential entrants other than the given buyer from group k] when the profile of entry probabilities for potential entrants is q. We set $P(N|q) = 0$ [resp. $P_k(N|q) = 0$] otherwise.
- $P(S|q^I) = \prod_{i \in S} q_i^I \cdot \prod_{i \in \mathcal{I} \setminus S} (1 - q_i^I)$ [resp. $P_i(S|q^I) = \prod_{i \in S} q_i^I \cdot \prod_{i \in \mathcal{I}_{-i} \setminus S} (1 - q_i^I)$] denote the probability of the realization $S \subseteq \mathcal{I}$ [resp. $S \subseteq \mathcal{I}_{-i}$] for the set of incumbents [the set of

incumbent except incumbent i] entering the mechanism when the profile of entry probabilities for the incumbents is q^I.

- $\Lambda_{N,S}(m, X; \sigma(m))$ denote the expected (interim) utility of the seller when the set of buyers entering the mechanism m consists of the profile of potential entrants N and the subset S of the incumbents, and when bidders follow the bidding profile $\sigma(m)$.

- $V_{k,N,S}(m; \sigma(m))$ [resp. $V^I_{i,N,S}(m; \sigma(m))$] denote the expected (interim) utility of a potential entrant from group k [resp. of incumbent i] when the set of buyers entering the mechanism m consists of the profile of potential entrants N with $n_k \geq 1$ [resp. $N \in \mathbb{N}^K$] together with the subset S of incumbents, and when buyers follow the bidding profile $\sigma(m)$.

- $u(q, q^I, m, X; \sigma(m)) = \sum_{N \in \mathbb{N}^K} \sum_{S \subseteq \mathcal{I}} P(N|q) \cdot P(S|q^I) \cdot \Lambda_{N,S}(m, X; \sigma(m))$ denote the expected (ex ante) utility of the seller with valuation X in the mechanism m when the profile of entry probabilities is q for potential entrants and q_I for incumbents and when buyers follow the bidding profile $\sigma(m)$.

- $u_k(q, q^I, m; \sigma(m)) = \sum_{N \in \mathbb{N}^K} \sum_{S \subseteq \mathcal{I}} P_k(N|q) \cdot P(S|q^I) \cdot V_{k,N_{+k},S}(m; \sigma(m))$] [resp. $u^I_i(q, q^I, m; \sigma(m)) = \sum_{N \in \mathbb{N}^K} \sum_{S \subseteq \mathcal{I}_{-i}} P(N|q) \cdot P_i(S_{-i}|q^I,) V_{i,N,S}(m; \sigma(m))$]] denote the expected (ex ante) utility of a group k buyer [resp. of incumbent i] in the mechanism m when the profile of entry probabilities is q for entrants and q_I for incumbents and when buyers follow the bidding profile $\sigma(m)$.

Note that in general the post-entry signal structure (i.e. the distribution of s_P and s for any pair N, S) intervenes in the computation of the above expectations. However, somewhat surprisingly, it does not play any role in our analysis so that there is no need to be explicit about this.

Finally, to present some of our results it is also convenient to define

- $W_{N,S}(m, X; \sigma(m))$ as the expected (interim) gross welfare (i.e. the expectation of the sum of all agents' valuations weighted by their probability to win[11] the good) conditional on the valuation X of the seller, the participation of the profile N of potential entrants and the subset S of incumbents, when the mechanism m is proposed and bidders follow the strategy $\sigma(m)$.

- $\Phi_{N,S}(m, X; \sigma(m))$ as the corresponding expected (interim) revenue of the seller, i.e.

$$\Phi_{N,S}(m, X; \sigma(m)) := W_{N,S}(m, X; \sigma(m)) - \sum_{k=1}^{K} n_k \cdot V_{k,N,S}(m; \sigma(m)) - \sum_{i \in S} V^I_{i,N,S}(m; \sigma(m)). \quad (1)$$

We say that the seller is a (pure) revenue-maximizer if her objective coincides with revenues, i.e., $\Lambda_{N,S}(m, X; \sigma(m)) = \Phi_{N,S}(m, X; \sigma(m))$.

[11]For the seller, it corresponds to *keeping* the good.

In the sequel, we will assume that bidders use undominated strategies. In particular, when the mechanism m is a second-price auction, buyers bid their valuation. To alleviate notation, whenever bidders have a (weakly) dominant strategy in mechanism m, we drop the reference to the strategy profile $\sigma(m)$ as bidders are assumed to use their (weakly) dominant strategy. In particular, $W_{N,S}(m_X^{ESP}, X)$ denotes the expected (gross) welfare in the (efficient) second-price auction in which the reserve price is set at the seller's valuation X, the set of participants is N for entrants and S for the incumbents, and bidders bid their valuations. Clearly, we have

$$W_{N,S}(m, X; \sigma(m)) \leq W_{N,S}(m_X^{ESP}, X) \tag{2}$$

for any profile N of entrants, any subset S of incumbents, any $m \in \mathcal{M}$ and any strategy profile $\sigma(m) \in \Sigma(m)$, since the efficient second-price auction puts the good into the hands of the participant (including the seller) who values it most with probability 1.

How many buyers of a given group k enter a mechanism is determined by an equilibrium condition reflecting some arbitrage between entering the given auction or staying out (thereby saving the entry cost). To define the equilibrium formally, we introduce for each $k \in \mathcal{K}$ a binomial parameter function $\widehat{q}_k : \mathcal{M} \to [0,1]$, where $\widehat{q}_k(m)$ stands for the entering probability of buyers from group k when the mechanism m is proposed. Similarly, we introduce for each $i \in \mathcal{I}$ the function $\widehat{q}_i^I : \mathcal{M} \to [0,1]$ to describe the probability with which incumbent i enters the various mechanisms $m \in M$. An equilibrium is defined as:

Definition 1 *For a given set of possible mechanisms $\mathcal{M} \subseteq \mathcal{M}^*$, an equilibrium with endogenous entry is defined as a strategy profile $(\widehat{m}, (\widehat{q}_k)_{k\in\mathcal{K}}, (\widehat{q}_i^I)_{i\in\mathcal{I}}, \widehat{\sigma})$, where $\widehat{m} \in \mathcal{M}$ stands for the seller's chosen mechanism, $\widehat{q}_k : \mathcal{M} \to [0,1]$ [resp. $\widehat{q}_i^I : \mathcal{M} \to [0,1]$] describes the entry probability of group k buyers [resp. the incumbent i] in the various possible mechanisms $m \in \mathcal{M}$, and $\widehat{\sigma}(m) \in \Sigma(m)$ describes the bidding profile of the bidders in $m \in \mathcal{M}$ such that*[12]

1. *(Utility maximization for the seller)*

$$\widehat{m} \in Arg \max_{m\in\mathcal{M}} \; u(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m)). \tag{3}$$

2. *(Utility maximization for group k buyers at the entry stage, for any $k \in \mathcal{K}$) for any $m \in \mathcal{M}$,*

$$\underset{resp.\ =\binom{0}{1}}{\widehat{q}_k(m) \;\in (0,1)} \Longrightarrow u_k(\widehat{q}(m), \widehat{q}^I(m), m; \widehat{\sigma}(m)) \underset{resp.\ \binom{\leq}{\geq}}{=} C_k. \tag{4}$$

[12]It should be noted that the profile of entry probabilities $(q(m), q^I(m))$ is defined irrespective of whether the mechanism m is offered in equilibrium, i.e. also for $m \neq \widehat{m}$. It is determined to ensure that when buyers consider whether to enter or not an auction, those decisions should always follow equilibrium behavior. This specification of profiles of entry probabilities (covering also non-chosen formats) is a simple way to capture trembling hand refinements, which are needed to rule out non-meaningful equilibria (in which suboptimal mechanisms would be offered being supported by irrational beliefs about the entry profiles attached to other mechanisms).

3. (Utility maximization for incumbent i at the entry stage, for any $i \in \mathcal{I}$) for any $m \in \mathcal{M}$,

$$\widehat{q}_i^I(m) \underset{resp. =\binom{0}{1}}{\in (0,1)} \Longrightarrow u_i^I(\widehat{q}(m), \widehat{q}^I(m), m; \widehat{\sigma}(m)) \underset{resp. \left(\substack{\leq \\ \geq}\right)}{=} C_i^I. \tag{5}$$

4. (Equilibrium conditions at the bidding stage) in any mechanism $m \in \mathcal{M}$, bidders are using undominated strategies. Furthermore, when the seller chooses the mechanism $\widehat{m}$, the bidding profile $\widehat{\sigma}(\widehat{m})$ forms a Bayes-Nash equilibrium given the profile of entry probabilities $(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m}))$.[13]

Condition (3) implies that the seller is required to pick a mechanism which maximizes her objective given the profile of entry probabilities $\widehat{q}(m) := (\widehat{q}_1(m), \ldots, \widehat{q}_K(m))$ and $\widehat{q}^I(m) := (\widehat{q}_1^I(m), \ldots, \widehat{q}_I^I(m))$, and the equilibrium bidding profile $\widehat{\sigma}(m)$ attached to any mechanism m. Condition (4) implies that whatever the mechanism and for each group $k \in \mathcal{K}$, either buyers use a strictly mixed entry strategy and the mechanism delivers an expected equilibrium utility of C_k to group k buyers, or buyers never enter and the corresponding expected payoff of a group k buyer is lower than C_k, or finally buyers enter with probability one and the corresponding expected payoff of a group k buyer is larger than C_k. Condition (5) is the analog of Condition (4) for incumbents. The difference between incumbents and potential entrants is that our equilibrium notion imposes implicitly that potential entrants inside a given group should behave in the same way, while such a restriction has no bite on incumbents.

The following assumption will guarantee that it is not an equilibrium for all entrants of group k to enter with probability 1 in the efficient auction.

Assumption A 1 *For any group $k \in \mathcal{K}$, $E_Z[\int_X^\infty [F_k(x|z)]^{\mathcal{N}_k - 1} \cdot (1 - F_k(x|z))dx] < C_k$.*

The left-hand-side in the above condition corresponds to the expected payoff in the efficient second-price auction of a group k buyer if all buyers from group k but no other buyer enter the auction. This is an upperbound on what a group k buyer can expect in the efficient second-price auction if all buyers from group k enter (and irrespective of what the entry decisions of other buyers are). Thus, requiring A1 is satisfied ensures that $\widehat{q}(m_X^{ESP}) \in [0,1)^K$. A1 is satisfied if $C_k \cdot \mathcal{N}_k$ is large enough for all k. In particular, a sufficient condition for A1 is that $C_k \cdot \mathcal{N}_k > \overline{x} - X$. For a given C_k, A1 holds if the total number of potential entrants in group k is large enough. In particular this will always be the case in the Poisson model analyzed in Section 5 which is the limit model where the $\mathcal{N}_k$s go to infinity.[14]

Comment: Observe that our setup allows for the possibility that $F_k = F_{k'}$ and/or $C_k = C_{k'}$ for $k \neq k'$. Since we make no assumption regarding how the entry probability of one group should

[13]At the bidding stage, we require equilibrium behavior only on the equilibrium path by contrast with the entry stage. We do not impose equilibrium behavior out of the equilibrium path in order to avoid the following little twist: With continuous games, equilibrium existence is not guaranteed so that the condition of Bayes-Nash equilibrium can be imposed only to the extent that an equilibrium exists.

[14]Having homogenous buyers inside a group plays also in favor of A1: in particular in case as in McAfee (1993) where each valuation distribution is concentrated at a point, then the left-hand-side is null for any $\mathcal{N}_k \geq 2$.

relate to that of another, the symmetry we impose within groups can be alleviated by splitting groups into several subgroups of identical characteristics. Note however that the splitting of a group into several subgroups makes it harder to satisfy A1 for each subgroup.

For a given mechanism $m \in \mathcal{M}$ and a given strategy profile $\sigma(m) \in \Sigma(m)$, we let

$$M(m;\sigma(m)) := \Big\{(q, q^I) \in [0,1]^{K+I} \text{ s.t. Conditions } (4) \text{ and } (5) \text{ hold for } m, \text{ and all } k \in \mathcal{K},\ i \in \mathcal{I}\Big\}$$

denote the set of profiles of entry probabilities that are compatible with equilibrium behavior. Observe that $M(m;\sigma(m))$ is non-empty.[15]

2.4 Related models in the literature

The case in which there are no incumbents ($I = 0$), entrants are homogenous ($K = 1$), the mechanism lies in $\mathcal{M}^r_{SP}$ and ex post signals correspond to bidders learning their own valuation, coincides with Levin and Smith's (1994) basic specification. In various parts of their paper, Athey et al. (2011, 2013) consider within the classical independent private value framework either the case with incumbents ($I \geq 1$) and homogenous entrants ($K = 1$) or no incumbents ($I = 0$) and two groups of entrants ($K = 2$). Concerning the set of mechanisms used in the counterfactual exercises, Athey et al. (2011, 2013) consider various sets of standard instruments, in particular entailing some form of direct discrimination (set-asides, bid subsidies) or some form of indirect discrimination (first- versus second-price auctions). It should be stressed that our informational assumptions contrast with those usually made (for example in Myerson, 1981) in which it is typically the case that the only signals received by bidders are their own valuations for the good while the seller does not receive any signal about bidders' valuations. Another degree of generality of our model is that we do not put any restriction on the number of groups. The group structure can thus capture the idea of pre-entry signals about valuations as in Roberts and Sweeting (2012) or even the limit case where potential entrants know their valuation ex ante as in McAfee (1993).[16] In our analysis, we assume that the seller cannot extract fees from the buyers after they have made their entry decision but before they have learned extra information about their valuation, an assumption which plays a role only in Section 4. In cases in which information acquisition would come after the entry decision and the seller could identify the group of any entrant, the seller could extract entirely the potential rent to be left to the buyer and thus induce the optimal entry rates through well adjusted fees.[17]

[15] Since there always exists a (group-symmetric) equilibrium in a game with a finite number of agents (where agent in the same group are symmetric) and finite action spaces (here the binary entry decisions), non-emptyness follows.

[16] Nevertheless, it does not capture models where the cost of entry is heterogenous among potential entrants as in Krasnokutskaya and Seim (2011) who consider a model à la Levin and Smith (1994) with two groups of buyers and heterogenous entry costs.

[17] By contrast, we stress that we do not impose that the seller observes the group k from which a buyer comes so that even if the seller could charge fees right after the entry decision but before the buyer can acquire further information, then it is not clear by the previous argument that an efficient non-discriminatory device would be optimal.

It is well-known that even in the simplest environments (e.g. with $K = 1$) there exist many asymmetric equilibria, in particular equilibria in which some potential entrants enter the auction with probability 1 and other potential entrants stay out. Such asymmetric equilibria are somehow similar to the equilibria arising in models with sequential entry (see Engelbrecht-Wiggans, 1993). While a limitation of our analysis, we see several rationales for our symmetry assumption. First, it is not clear by which mechanism, potential entrants coming from the same group would manage to coordinate their entry decisions in such an asymmetric way. Second, as already noted, a given group can always be split into several subgroups with identical characteristics including possibly subgroups consisting of a single buyer who could then be thought of as incumbents according to the above terminology. Last, the empirical literature dealing with structural models with entry makes also such symmetry restrictions meaning that our results can shed some new light on the forces that drive their counterfactual exercises.

Although our model involves a single seller, it can be viewed as a reduced form model of richer models of competition between many (possibly heterogenous) sellers and many (possibly heterogenous) buyers in which the cost of entry of a group k buyer corresponds to the expected utility such a buyer could obtain elsewhere as in McAfee (1993). A key aspect is that the entry costs C_k and C_i^I do not depend on the mechanism proposed by the seller. This exogeneity restriction reflects that each seller has no market power.

3 A general non-discrimination result

In order to present our first non-discrimination result more compactly, we consider here the presence of incumbents and we assume that the objective of the seller is her revenue augmented by the sum of the profits of the incumbents. The main application we have in mind, which is a special case of this, is when there are no incumbents and the seller seeks to maximize revenues. Formally,

Assumption A 2 $\Lambda_{N,S}(m, X; \sigma(m)) = \Phi_{N,S}(m, X; \sigma(m)) + \sum_{i \in S} \left[V_{i,N,S}^I(m; \sigma(m)) - C_i^I\right]$.

Combining (1) with A2, we obtain that

$$u(q, q^I, m, X; \sigma(m)) = \sum_{N \in \mathbb{N}^K} \sum_{S \subseteq \mathcal{I}} P(N|q) \cdot P(S|q^I) \cdot W_{N,S}(m, X; \sigma(m)) - \sum_{k=1}^{K} q_k \mathcal{N}_k \cdot u_k(q, q^I, m; \sigma(m)) - \sum_{i=1}^{I} q_i^I \cdot C_i^I. \tag{6}$$

3.1 Some fundamental observations

For a given mechanism m proposed by the seller with valuation X, for a given bidding strategy profile $\sigma(m)$ and when the entry probability of group k entrants [resp. of incumbent i] is q_k for $k \in \mathcal{K}$ [resp. q_i^I for $i \in \mathcal{I}$], we define the net expected (ex ante) welfare (the welfare from which entry costs are deducted) as

$$NW(q,q^I,m,X;\sigma(m)) := \sum_{N\in\mathbb{N}^K}\sum_{S\subseteq\mathcal{I}} P(N|q)\cdot P(S|q^I)\cdot W_{N,S}(m,X;\sigma(m)) - \sum_{k=1}^{K} q_k\mathcal{N}_k\cdot C_k - \sum_{i=1}^{I} q_i^I\cdot C_i^I \quad (7)$$

and we let $J(m,X;\sigma(m)) := \operatorname{Arg\,max}_{(q,q^I)\in[0,1]^{K+I}} NW(q,q^I,m,X;\sigma(m))$.

An upperbound on the seller's payoff

Combining the expressions (6) and (7) with the utility maximization conditions (4) for potential entrants, we obtain for any mechanism m and any strategy profile $\sigma(m)\in\Sigma(m)$ that

$$u(q,q^I,m,X;\sigma(m)) \leq NW(q,q^I,m,X;\sigma(m)) \quad (8)$$

for any entry probabilities that are compatible with equilibrium behavior, i.e. $(q,q^I)\in M(m;\sigma(m))$ and where the inequality stands as an equality if we have furthermore $q\in[0,1)^K$. Besides, when A1 holds, the probability of entry must be less than 1 in all groups of entrants in the efficient second-price auction (formally $M(m_X^{ESP})\subseteq[0,1)^K\times[0,1]^I$). As a corollary, we obtain that in equilibrium the seller's revenue coincides with the net welfare in the efficient second-price auction, namely

$$u(\widehat{q}(m_X^{ESP}),\widehat{q}^I(m_X^{ESP}),m_X^{ESP},X) = NW(\widehat{q}(m_X^{ESP}),\widehat{q}^I(m_X^{ESP}),m_X^{ESP},X). \quad (9)$$

We obtain from (8) that at a given equilibrium, the seller's expected utility is bounded from above by the first-best solution,[18] which is defined as the solution to the maximization program

$$\max_{(q,q^I)\in[0,1]^{K+I},m\in\mathcal{M},\sigma(m)\in\Sigma(m)} NW(q,q^I,m,X;\sigma(m)) \quad (10)$$

in which the entry probabilities (q,q^I) are not constrained in any way. We say that an equilibrium (or a mechanism m) implements the first-best when the expected utility of the seller coincides with this upperbound (when m is proposed by the seller).

As we will show, the efficient second-price auction m_X^{ESP}, together with a vector of entry probabilities (q,q^I) that is compatible with equilibrium behavior for m_X^{ESP} implements the first-best. This will in turn guarantee that such a profile of mechanism and entry probabilities are part of an equilibrium, since no deviation of the seller could ever induce a higher expected utility for her. It will also establish that the outcome of such an equilibrium is the one most preferred by the seller among all equilibrium outcomes (in case several equilibrium outcomes could arise).

To see that m_X^{ESP} may allow to implement the first-best, we make the following two simple observations, which are well known properties of the efficient second-price auction.

Efficiency given entry probabilities

First, given that m_X^{ESP} allocates the good to the agent with highest valuation (including the

[18]It should be mentioned that our notion of first-best assumes implicitly that entrants of the same group behave symmetrically. This is the notion adapted to deal with the notion of equilibrium as introduced in Section 2.

seller), it is clear that for any given entry probabilities (q, q^I), no other mechanism can induce a higher value of net welfare. Formally, from (2), we have that for any $(q, q^I) \in [0,1]^{K+I}$ and any $m \in \mathcal{M}$ and any $\sigma(m) \in \Sigma(m)$,

$$NW(q, q^I, m, X; \sigma(m)) \leq NW(q, q^I, m_X^{ESP}, X).$$

As a result of the previous observations, letting $(q, q^I) \in J(m_X^{ESP}, X)$ be the optimal entry probabilities, if we could show that these can be part of equilibrium entry probabilities for m_X^{ESP}, i.e. $(q, q^I) \in M(m_X^{ESP})$, then we would be able to conclude that the upperbound (10) can be reached in equilibrium.

Optimality of equilibrium entry probabilities

A key property of the efficient second-price auction is that for any given set of competitors (N, S), the expected payoff received by a buyer deciding to enter coincides with the contribution of his entry to the expected welfare. This fundamental property can be stated formally as

$$W_{N_{+k},S}(m_X^{ESP}, X) - W_{N,S}(m_X^{ESP}, X) = V_{k,N_{+k},S}(m_X^{ESP}) \tag{11}$$

for any entrant of group $k \in \mathcal{K}$, and similarly $W_{N,S}(m_X^{ESP}, X) - W_{N,S_{-i}}(m_X^{ESP}, X) = V_{i,N,S}^I(m_X^{ESP})$ for any incumbent $i \in S$. As each buyer gets the incremental surplus he generates in m_X^{ESP}, simple algebra leads to (more details are given in the Appendix –Proof of Lemma 3.1):

$$\frac{\partial NW(q, q^I, m_X^{ESP}, X)}{\partial q_k} = \mathcal{N}_k \cdot \left[u_k(q, q^I, m_X^{ESP}) - C_k\right] \tag{12}$$

for group k entrants and similarly $\frac{\partial NW(q,q^I,m_X^{ESP},X)}{\partial q_i^I} = u_i^I(q, q^I, m_X^{ESP}) - C_i^I$ for any incumbent i.

Combining (12) with the first-order conditions for the optimality of (q, q^I) with respect to potential entrants' entry decisions and then the analog properties for incumbents, we obtain that any pair $(q, q^I) \in J(m_X^{ESP}, X)$ is compatible with equilibrium behavior.

To summarize the above observations, we may state:

Lemma 3.1 *Assume A1 and A2. We have* $J(m_X^{ESP}, X) \subseteq M(m_X^{ESP}) \subseteq [0,1)^K \times [0,1]^I$.

For allocation problems (and beyond the assignment problem considered here), it is well-known that in private value setups when the pivotal mechanism[19] (in which an efficient alternative is chosen and agents pay the welfare loss their presence imposes on others) is preceded by a stage in which agents are making private pre-participation investments (i.e. that influence only their own type), then any profile of investments that maximizes the net welfare (net of the investments

[19]It is also called the Clarke mechanism or the VCG mechanism in the literature (see Mas-Colell et al., 1995). The efficient second-price auction (also called the Vickrey auction) is a special case of the pivotal mechanism for single good allocation problems.

costs) is an equilibrium.[20] From a broader perspective, the games that govern the choices of pre-investment strategies can be seen as potential games (Monderer and Shapley, 1996) in which the potential function is equal to the net welfare. In our environment, entry decisions can be viewed as a specific form of pre-participation investments (not entering can be viewed equivalently as inducing a null valuation), thereby explaining why the efficient entry probabilities can arise in equilibrium in m_X^{ESP}.[21] It should be mentioned that in the context of potential games, some scholars have suggested that players would coordinate on the equilibrium that maximizes the potential function globally (see Haufbauer and Sorger (1999) and Carbonell-Nicolau and McLean (2014)). Such a selection corresponds to the one considered below.

3.2 The non-discriminatory equilibrium and its properties

The previous arguments establish that it is an equilibrium for the seller to choose m_X^{ESP} and for the buyers to chose an entry profile that maximizes the net welfare. Thus, there is an equilibrium that implements the first-best. Observe that given the above discussion (related to the upperbound), this is the equilibrium most preferred by the seller.

Proposition 3.2 *Assume that $m_X^{ESP} \in \mathcal{M}$, and that A1 and A2 hold.*[22] *There exists an equilibrium in which the seller proposes the efficient second-price auction m_X^{ESP} and the equilibrium entry probabilities belong to the set $J(m_X^{ESP}, X)$. Such an equilibrium implements the first-best.*

We obtain a partial converse if we make the following refinement:

PG-refinement: If the mechanism proposed by the seller induces a potential game among a set of buyers, the equilibrium played maximizes the corresponding potential function.

Since the efficient second-price auction is a potential game with the total net welfare as the potential function, the PG-refinement guarantees that $(\widehat{q}(m_X^{ESP}), \widehat{q}^I(m_X^{ESP})) \in J(m_X^{ESP}, X)$ and thus that any equilibrium implements the first-best (otherwise posting m_X^{ESP} would be a profitable deviation). We also establish that any equilibrium that implements the first-best is "equivalent" to the one arising with the efficient second-price auction where equivalence between strategy profiles is formally defined by:

Definition 2 *We say that two strategy profiles $(m, \{q_k\}_{k\in\mathcal{K}}, \{q_i^I\}_{i\in\mathcal{I}}, \{\sigma(m)\}_{m\in\mathcal{M}})$ and $(\widetilde{m}, \{\widetilde{q}_k\}_{k\in\mathcal{K}}, \{\widetilde{q}_i^I\}_{i\in\mathcal{I}}, \{\widetilde{\sigma}(m)\}_{m\in\mathcal{M}})$ are equivalent if the profile of entry probabilities at the mechanism proposed by the seller are the same, namely $q(m) = \widetilde{q}(\widetilde{m})$ and $q^I(m) = \widetilde{q}^I(\widetilde{m})$, and*

[20] See Rogerson (1992) and more recently, Bergemann and Välimäki (2002) in a context with information acquisition, Arozamena and Cantillon (2004) when buyers can upgrade their valuation distribution and Stegeman (1996) in auctions with participation costs. The key point in all those papers is that in the pivotal mechanism the maximization program faced by each agent corresponds to the maximization of the net welfare.

[21] Technically, there is also a small twist with the standard theory of potential games since we constrain buyers from the same group to behave symmetrically, thereby explaining why $u_k(q, q^I, m_X^{ESP}) - C_k$ is multiplied by $\mathcal{N}_k$ in the right hand-side of (12).

[22] Assumption A1 is a sufficient condition that is easy to interpret. However, it is much stronger than needed for Proposition 3.2. Fundamentally, for the existence of an equilibrium without discrimination that implements the first-best, the condition that is needed is $J(m_X^{ESP}, X) \cap ([0,1)^K \times [0,1]^I) \neq \emptyset$, namely that there exists an entry profile that implements the first-best and such that entrants use mixed-strategies so that their rents are inelastic.

if for any profile of bidders (N, S) that occurs with positive probability,[23] *then the good is assigned in the same way with probability one (which implies in particular that $W_{N,S}(m, X; \sigma(m)) = W_{N,S}(\widetilde{m}, X; \widetilde{\sigma}(m))$).*

We have:

Proposition 3.3 *Assume that $m_X^{ESP} \in \mathcal{M}$, and that A1 and A2 hold. Any equilibrium that satisfies the PG-refinement is equivalent to the equilibrium that we have exhibited in Proposition 3.2 and implements thus the first-best.*

Let us briefly discuss the more general setup in which buyers receive a pre-entry public signal s_Z so that entry decisions are now made conditional on this signal. The generalization of the maximization program (10) which provides an upperbound on the seller's revenue is now

$$\max_{(q,q^I):\mathcal{S}_Z \to [0,1]^{K+I}, m \in \mathcal{M}, \sigma(m) \in \Sigma(m)} E_{s_z \in \mathcal{S}_Z}[NW(q, q^I, m, X; \sigma(m)|s_z)], \tag{13}$$

in which the entry probabilities function (q, q^I) are not constrained in any way. If Assumption 1 holds conditional on any realization of the signal s_Z,[24] then Proposition 3.2 carries over and the efficient second-price auction implements the first-best. A simple corollary can then be obtained regarding the interest of providing more information to the buyers.

Corollary 3.4 *In the equilibria that implement the first-best, the seller is better off if buyers get more information before entry.*

Such a result trivially follows because the more information buyers have ex ante, the more finely they can condition their entry decisions on the true state of the world and thus the highest the solution of the maximization program (13). It follows that the seller is better off if she can commit to disclosing as much information as possible (assuming buyers coordinate on the net welfare maximizing equilibrium when the seller posts m_X^{ESP}).

It may be mentioned that we obtain a conclusion regarding information disclosure that is somehow similar to the one arising in the case of affiliated interdependent values (see the references to the "linkage principle" for example in Milgrom (2004) who recasts it as the "publicity effect"). Yet, despite the common conclusion, the channel through which the result is obtained is of a completely different nature: Here the ex ante information does not have any impact on the way buyers bid (since we are in a private value setting where bidding his valuation is a dominant strategy in second-price auctions) but rather on their entry decisions.

Further comments: 1) Our result that in the equilibrium most preferred by the seller, the seller chooses the efficient second-price auction has a "robust mechanism design" flavor insofar as the seller need not have any information about the primitives of the model (except her own valuation)

[23] Formally, $P(N|q(m)) \cdot P(S|q^I(m)) > 0$.

[24] Formally we mean that for any group $k \in \mathcal{K}$ and any $s_Z \in \mathcal{S}_Z$, $E_{Z|s_Z}[\int_X^{\infty} [F_k(x|z)]^{\mathcal{N}_k - 1} \cdot (1 - F_k(x|z))dx] < C_k$.

to make her choice of mechanism. Second, once in the mechanism, buyers have a dominant strategy, which implies that buyers' bidding strategies are robust to how beliefs about others' information and strategies are specified. It should be noted however that the entry probabilities required in equilibrium rely on more elaborate adjustments to the primitives of the model.[25] 2) Our analysis extends to the case in which the number of potential entrants in each group is no longer deterministic but stochastic provided that buyers have no private information about the realization of those variables and provided that an analog of A1 holds, i.e. an assumption which guarantees that for each group potential entrants never enter the mechanism with probability one. 3) Our results hold no matter what the correlations between buyers' valuations are, so in particular even in situations in which using devices à la Crémer-McLean (1988), the seller could extract fully the informational rents for a given participation profile. Of course, if the seller were to fully extract the surplus of the participants in the mechanism, no potential entrant would enter. What our analysis reveals more generally is that no profitable use can be made of such correlations. 4) It should be stressed that Proposition 3.2 does not say that there exists an equilibrium that implements the first-best for any ex post efficient mechanism. We have already illustrated this above when considering the possibility of correlation. To provide another illustration, consider that the seller's valuation lies strictly below the lower bound of buyers' valuation distributions. Then every second-price auction with a reserve price between zero and this lower bound would achieve an ex post efficient allocation. Yet, if $J(m_X^{ESP}, X) \cap \{(q, q^I) \in [0,1)^K \times [0,1]^I | q = (0, \ldots, 0)\} = \emptyset$, then only the reserve price set at the seller's valuation is able to implement the first-best: any other reserve price would fail to induce efficient participation rates.[26,27] 5) Let us discuss informally the more general class of models in which the potential entrants inside a given group may be heterogenous before entry either because their entry costs are heterogenous (while assuming that this cost is a private signal distributed from the same distribution) or because they receive some private signal about their valuation (see e.g. Krasnokutskaya and Seim (2011) and Roberts and Sweeting (2012, 2013)). Let us also assume that this ex ante signal of a given potential entrant is of no help to assess the other buyers' valuation. If these underlying signals do not enter the bidders' payoff functions neither directly nor indirectly by providing some information on valuations, then this can be viewed as a way to purify the equilibrium with mixed strategies that we have exhibited in Proposition 3.2. If however the underlying signals do not reduce to white noises, then our analysis would be affected, since it opens the door to the possibility that the rents received by the entrants would depend on the mechanism that is proposed by the seller. The set of equilibria with endogenous entry would then be more complicated to analyze. Nevertheless, if those rents are

[25] Equilibrium analysis in a framework that relaxes the rationality assumption by considering that some buyers have (analogy-based) coarse expectations in the vein of Jehiel (2005) is the subject of Jehiel and Lamy (2015).

[26] This is a corollary of Lemma 7.1 (used in the proof of Proposition 6.1) which shows that two distinct second-price auctions can not lead to the same entry profile where some potential entrants chose to enter with positive probability.

[27] This discussion points the finger on the discrepancy between our pivotal mechanism (referred to as the efficient second-price auction in our single good environment) and the pivotal mechanism as defined by Krishna and Perry (1998). The latter leaves no rents to the buyers with the lowest type.

small in expectation (which will be the case in particular if the underlying heterogeneity has little payoff consequences in each group), the loss of the seller is small when she proposes m_X^{ESP} while inducing the welfare-maximizing entry probabilities (which is still compatible with equilibrium behavior since this ingredient relies only on the private value assumption). More formally, if the sum of the expected profits (net of the entry costs) of the potential entrants is bounded from above by ϵ in m_X^{ESP} and with a profile of entry probabilities that maximizes the net total expected welfare, then the loss of the seller when she chooses m_X^{ESP} is also bounded from above by ϵ. This is so because the seller's revenue is still bounded from above by the net total expected welfare.

We present below a detailed description of the literature which helps to better locate our contribution. We also discuss the possibility of multiple equilibria and its consequence on discrimination (to which we will come back in Section 5). Those readers willing to see our next results can jump into Section 4.

3.3 Relation with the previous literature

Proposition 3.2 can be viewed as generalizing the main insight obtained by Levin and Smith (1994) and McAfee (1993) under more constrained informational assumptions: There exists an equilibrium which results in an ex post efficient allocation. Proposition 3.2 says also a bit more since such a "good" equilibrium appears to be also ex ante optimal, an insight that has not been previously mentioned in McAfee's (1993) kinds of model. The logic of our non-discrimination result comes from the combination of two simple ingredients. On the one hand, whatever the chosen mechanism the seller's expected revenue is smaller than the net total welfare, and in the efficient second-price auction, potential entrants exhaust their rents through entry so that the the seller's objective coincides with net welfare. On the other hand, the efficient second-price auction is such that ex ante efficient entry probabilities constitute an equilibrium profile, which is so because the payoff of each bidder corresponds to his contribution to the welfare in such a mechanism under private values. Combined together, these insights imply that one equilibrium of the auction game with endogenous entry requires that there be no discrimination and also the stronger result that the seller's most preferred equilibria are all equivalent to such an equilibrium without discrimination. This decomposition has led us to obtain our result in great generality, which should be contrasted with the more computational approach developed in McAfee (1993), Peters (1997) and Peters and Severinov (1997).

The literature has often classified auction models with endogenous entry according to whether buyers have no private information before entry (McAfee and McMillan 1987, Engelbrecht-Wiggans 1993, Levin and Smith 1994) or buyers know perfectly their valuation before choosing to incur the entry costs (Samuelson 1985, McAfee 1993, Stegeman 1996). Our analysis reveals that this is not the most relevant classification, and, in fact, our results also apply to intermediate cases in which buyers have some ex ante private information which they refine after entry.[28] Proposition

[28]It is instructive to delineate why Samuelson (1985) and McAfee (1993), two seemingly equivalent models, lead

3.2 covers also environments with incumbents provided that their rents are *fully* internalized by the seller. Whenever incumbents (resp. potential entrants) correspond to domestic (resp. foreign) firms, note that this result strongly contrast with McAfee and McMillan (1989) who obtain that the optimal mechanism involves favoritism towards domestic firms independently of the respective strengths of the different firms. In Section 4, we will push the analysis further by covering also the cases in which the seller may not (or only partially) internalize the profits of the incumbents.

3.4 Inefficient equilibria and the uniqueness problem

The analysis so far has not considered whether there could be multiple equilibria at the entry stage (assuming we do not impose the PG-refinement). A possibility is that when the seller uses the efficient second-price auction, there would be other entry equilibria and that some of them would be welfare-inefficient. But, if equilibrium is inefficient at the entry stage, there is no guarantee anymore that the seller would not prefer using another auction format possibly inducing some discrimination so as to gain with respect to the entry decisions. Specifically, the question is whether there could exist some $(\widetilde{q}, \widetilde{q}^I) \in M(m_X^{ESP})$ such that $(\widetilde{q}, \widetilde{q}^I) \notin J(m_X^{ESP}, X)$. A simple sufficient condition that guarantees that this never occurs is that the function $(q, q^I) \to NW(q, q^I, m_X^{ESP}, X)$ is (globally) concave, a condition we will establish in Section 5 in the limit when the number of potential entrants in each group goes to infinity.[29] In such a case, we have the stronger observation that all equilibria are equivalent to the equilibrium without discrimination that we have exhibited in Proposition 3.2. To the best of our knowledge, our paper is the first to establish such a strong form of non-discrimination result.

Comment: The aforementioned literature about auctions preceded by a pre-participation investment stage has also shown that in the pivotal mechanism there may exist other equilibrium investment profiles that are not global optima of the welfare but only local optima. For example, in second-price auctions with participation costs, Stegeman (1996) exhibits an example with symmetric bidders where the symmetric equilibrium is inefficient. In the literature about affirmative action (see Fang and Moro (2011) for a recent survey), it is often the case that discrimination

to completely different insights, in particular regarding the use of reserve prices. Contrary to McAfee (1993), the optimal mechanism in Samuelson's (1985) setup requires thus some discrimination between bidders if the latter are ex ante asymmetric and also that the seller keeps more often the good than efficiency would dictate (see Celik and Yilankaya, 2009). A key aspect in McAfee (1993) is that each seller has no market power: She takes as given the function that maps buyers' valuation to their expected utility. By contrast, in Samuelson's (1985) setup, only the expected rents of the buyer with the type that is indifferent between participating or not in the mechanism is fixed ex ante: The expected rent of the participants do depend on the mechanism proposed which leads to a trade-off between maximizing allocative efficiency and minimizing buyers' rents and thus to discrimination insights à la Myerson (1981).

[29]To appreciate how surprising our concavity result can be, note that $\frac{\partial^2 NW(q,q^I,m,X;\sigma(m))}{\partial^2 q_i^I} = 0$ (and similarly we would have $\frac{\partial^2 NW(q,q^I,m,X;\sigma(m))}{\partial^2 q_k} = 0$ if $\mathcal{N}_k = 1$) for any $m \in \mathcal{M}$. Furthermore two zeros on the diagonal of the Hessian matrix prevent concavity (More precisely, this is so except possibly in the case where the corresponding off-diagonal terms are also null (i.e. $\frac{\partial^2 NW(q,q^I,m,X)}{\partial q_i^I \partial q_{i'}^I} = 0$ when $i \neq i'$). It is easy to see that it should not be the case generically in our setup). Putting several potential buyers in a group (and given that we assume that they behave symmetrically) has a concavification effect: this is what leads to our uniqueness result in Section 5.

is justified by some miscoordination problems, i.e. when the economy is stuck in a bad equilibrium.[30] As our analysis in Section 5 reveals, no such motive for discrimination can be justified in the context of auctions with large sets of potential entrants.

4 Optimal discrimination in the presence of incumbents

4.1 The environment with incumbents

We now consider that the seller does not necessarily fully internalize the profits of the incumbents. In particular, we allow her to be a pure revenue maximizer. More generally, we allow her to internalize a positive share β_i^I (assumed to be less than 1) of the expected profits of incumbent i. Specifically, we assume:

Assumption A 3 $\Lambda_{N,S}(m, X; \sigma(m)) := \Phi_{N,S}(m, X; \sigma(m)) + \sum_{i \in S} \beta_i^I \cdot [V_{i,N,S}^I(m; \sigma(m)) - C_i^I]$ *with* $0 \leq \beta_i^I < 1$ *for any* $i \in \mathcal{I}$.

For government procurements, we have in mind that incumbents correspond to domestic firms and that a share of their profit is internalized by the government through taxation.

Our interest lies in understanding whether discrimination between incumbents and potential entrants is desirable in such a case and how the answer depends on whether incumbents are weaker or stronger than potential entrants (as measured by their respective CDFs). Our main insight is that incumbents should be discriminated against entrants no matter whether they are stronger or weaker than entrants and no matter which share of their profits is internalized by the seller. Moreover, we characterize the exact form of optimal discrimination in the vein of Myerson's (1981) analysis.

An essential difference with the previous section is that there is now a discrepancy between the seller's objective and the net total welfare, where the difference comes from the incumbents' rents. In the previous Section, the information structure of incumbents and of potential entrants (which are determinants of buyers' rents) did not play any role. By contrast, in this Section we have to impose some extra structure in order to be able to express in a tractable (and not degenerate) way the rents of the incumbents. Essentially, as in Myerson (1981), we make the assumption that the valuation of each incumbent is independent of any information held by his competitors. Formally,

Assumption A 4 *1) For each incumbent* $i \in \mathcal{I}$*: The distribution* $F_i^I(.|z)$ *does not depend on* z *and is denoted by* $F_i^I(.)$*. Furthermore,* $F_i^I(.)$ *is continuously differentiable on its supports* $[\underline{x}_i, \overline{x}_i]$ *with a density, denoted by* $f_i^I(.)$*, that is strictly positive. The vector gathering the signals received by the other buyers, the seller and also the extra signals received by incumbent* i *are assumed to be independent of incumbent* i*'s valuation. 2) For each* $i \in \mathcal{I}$*, the CDF* $F_i^I(.)$ *is regular, namely*

[30] E.g., Coate and Loury (1993) propose a competitive labor market model with non-contractible pre-investments (efforts made by workers) with two ex ante identical groups but with Pareto ranked equilibria, in particular where one group is stuck in a self-fulling bad reputation trap.

$x \to \frac{1-F_i^I(x)}{f_i^I(x)}$ *is strictly decreasing on* $[\underline{x}_i, \overline{x}_i]$*, and (to simplify the presentation) we also assume that* $\underline{x}_i - \frac{1-F_i^I(\underline{x}_i)}{f_i^I(\underline{x}_i)} \leq X$. *3) The seller observes the identity of the incumbents.*

Condition 1 of A4 ensures some independence of incumbents' valuation from any other information held by other buyers and is needed to avoid the full extraction results à la Crémer-McLean (1988). This is an assumption also made in Myerson (1981). Apart from the independence assumption, we also assume in Condition 3 that the seller can observe the identity of incumbents so that she can use direct discrimination (as in Myerson's work). Finally, Condition 2 simplifies the presentation of the result in that it avoids the use of ironing techniques. Note however that we make no assumption about how the private information of entrants is distributed nor about what the seller observes about potential entrants.

Some additional notation is required before we can state our main result. We let

- $V_i^I(x, m; q, q^I, \sigma(m))$ denote the expected utility of incumbent i conditional on having valuation x in the mechanism m when the profile of entry probabilities is (q, q^I) and when buyers follow the bidding profile $\sigma(m)$. Note that $u_i^I(q, q^I, m; \sigma(m)) = \int_{\underline{x}_i}^{\overline{x}_i} V_i^I(x, m; q, q^I, \sigma(m))dF_i^I(x)$.

- $x^S = \{x_i^S\}_{i \in S}$ denote the realization of the vector of valuations of the set of incumbents given that the set of incumbents who decide to enter the mechanism is S.

- $x^N = (x_1^N, \ldots, x_{|N|}^N)$ denote the realization of the vector of valuations of the potential entrants given that the profile of potential entrants who decide to enter the mechanism is N and with $|N| = \sum_{k=1}^K n_k$.

- $y^{N,S}$ denote the realization of the vector of all the signals of all the entrants N, the incumbents in $S \subseteq \mathcal{I}$ and the seller. Note that we can recover x^S and x^N from $y^{N,S}$.

- $y_{-i}^{N,S}$ denote the realization of the vector of all the signals of all the entrants N, the incumbents in $S \subseteq \mathcal{I}$ and the seller except the valuation of the incumbent $i \in S$. Thus, we have $y^{N,S} = (x_i^S, y_{-i}^{N,S})$.

- $G_{N,S}(.)$ [resp. $G_{-i,N,S}(.)$ with $i \in S$] denote the distribution of $y^{N,S}$ [resp. $y_{-i}^{N,S}$] for a given profile of potential entrants N and the set of incumbent S. From A4, we have for any $i \in S$

$$G_{N,S}(y^{N,S}) = F_i^I(x_i^S) \cdot G_{-i,N,S}(y_{-i}^{N,S}). \tag{14}$$

- $Q_{i,N,S}^I(y^{N,S}, m; \sigma(m))$ [resp. $Q_{j,N,S}(y^{N,S}, m; \sigma(m))$] denote the probability that the incumbent $i \in S$ [resp. the entrant $j \in \{1, \ldots, |N|\}$] receives the good in the mechanism m when bidders follow the strategies $\sigma(m)$ and when the realization of the signals is $y^{N,S}$.

- $Q_{0,N,S}(y^{N,S}, m; \sigma(m)) = 1 - \sum_{i \in S} Q_{i,N,S}^I(y^{N,S}, m; \sigma(m)) - \sum_{j=1}^{|N|} Q_{j,N,S}(y^{N,S}, m; \sigma(m))$, denote the corresponding probability that the seller keeps the good.

4.2 The virtual first-best as an upperbound on the seller's payoff

Combining the expression for the expected revenue (1) with standard calculations in the vein of Myerson (1981) allows us to express incumbents' rents as a function of their corresponding probability to win the good. After standard transformations, we obtain that for any mechanism $m \in \mathcal{M}$, the seller's objective in equilibrium is given by

$$
\begin{aligned}
u(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m)) = & \sum_{N \in \mathbb{N}^K} \sum_{S \subseteq \mathcal{I}} P(N|\widehat{q}(m)) \cdot P(S|\widehat{q}^I(m)) \cdot W_{N,S}^{virt}(m, X; \widehat{\sigma}(m)) \\
& - \sum_{k=1}^{K} \widehat{q}_k(m) \cdot \mathcal{N}_k \cdot u_k(\widehat{q}(m), \widehat{q}^I(m), m; \widehat{\sigma}(m)) \\
& - \sum_{i=1}^{I} (1 - \beta_i^I) \cdot \widehat{q}_i^I(m) \cdot V_i^I(\underline{x}_i, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m)) - \sum_{i=1}^{I} \beta_i^I \cdot \widehat{q}_i^I(m) \cdot C_i^I
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
W_{N,S}^{virt}(m, X; \widehat{\sigma}(m)) := \int \Bigg\{ & Q_{0,N,S}(y^{N,S}, m; \widehat{\sigma}(m)) \cdot X + \sum_{j=1}^{|N|} Q_{j,N,S}(y^{N,S}, m; \widehat{\sigma}(m)) \cdot x_j^N \\
& + \sum_{i \in S} Q_{i,N,S}^I(y^{N,S}, m; \widehat{\sigma}(m)) \cdot \left[x_i^S - (1 - \beta_i^I) \cdot \frac{1 - F_i^I(x_i^S)}{f_i^I(x_i^S)} \right] \Bigg\} dG_{N,S}(y^{N,S})
\end{aligned}
\tag{16}
$$

is referred to as the expected *virtual welfare*. The virtual welfare corresponds to the total expected welfare that would be obtained if, while keeping the valuations of potential entrants unchanged, the valuations of incumbents were replaced by their virtual valuations where the mapping between true and virtual valuations of incumbent i, for any $i \in \mathcal{I}$, is defined in a similar way as in Myerson (1981) by[31]

$$
x_i^{virt}(x) := x - (1 - \beta_i^I) \cdot \frac{1 - F_i^I(x)}{f_i^I(x)} \tag{17}
$$

for any $x \in [\underline{x}_i, \overline{x}_i]$ and $x_i^{virt}(x) := x$ for $x > \overline{x}_i$. In the sequel, when we refer to the virtual valuations, we mean the true valuation for a potential entrant or the seller and the virtual valuation as just defined for incumbents. Analogously to Section 3, we let

$$
NW^{virt}(q, q^I, m, X; \sigma(m)) := \sum_{N \in \mathbb{N}^K} \sum_{S \subseteq \mathcal{I}} P(N|q) \cdot P(S|q^I) \cdot W_{N,S}^{virt}(m, X; \sigma(m)) - \sum_{k=1}^{K} q_k \cdot \mathcal{N}_k \cdot C_k - \sum_{i=1}^{I} \beta_i^I \cdot q_i^I \cdot C_i^I \tag{18}
$$

denote the net expected (ex ante) virtual welfare and

$J^{virt}(m, X; \sigma(m)) := \operatorname{Arg\,max}_{(q,q^I) \in [0,1]^{K+I}} NW^{virt}(q, q^I, m, X; \sigma(m))$.

Combining the expressions (15) and (18) with the utility maximization conditions (4) for potential entrants and since for any incumbent i with $\widehat{q}_i^I(m) > 0$, the participation constraint at the auction stage is equivalent to $V_i^I(\underline{x}_i, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m)) \geq 0$, we obtain that for any $m \in \mathcal{M}$

$$
u(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m)) \leq NW^{virt}(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m)). \tag{19}
$$

[31] $x_i^{virt}(\cdot)$ is an increasing function on $[\underline{x}_i, \infty)$ thanks to our regularity assumption in A4. We let $[x_i^{virt}]^{-1}(\cdot)$ denote the corresponding inverse function.

Furthermore, if $\widehat{q}(m) \in [0,1)^K$ and $V_i^I(\underline{x}_i, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m)) = 0$ for any $i \in \mathcal{I}$, then the previous inequality stands as an equality. Consequently, the seller's expected objective can never outperform the solution of the maximization program

$$\max_{(q,q^I)\in[0,1]^{K+I}, m\in\mathcal{M}, \sigma(m)\in\Sigma(m)} NW^{virt}(q, q^I, m, X; \sigma(m)). \tag{20}$$

Similarly to Section 3, we say that an equilibrium (or a mechanism) implements the virtual first-best if the seller's expected objective reaches the upperbound characterized by (20). From (19), equilibria which implement the virtual first-best (if they exist) have another theoretical status, since they are the seller's most preferred equilibria.

4.3 The virtual pivotal mechanism

As we will show, this upperbound can be reached when the seller uses the *virtual pivotal mechanism*, denoted by $m_{\beta,X}^{v\text{-}piv}$, which is defined as the auction that assigns the good to the agent with the highest virtual valuation (including the seller) and has the winner (if any) pay the valuation that would make him match the second-highest virtual valuation (including the seller) while losing bidders do not pay anything. The formal definition is as follows.

Definition 3 *The virtual pivotal mechanism is the direct mechanism such that for any $N \in \mathbb{N}^K$, $S \subseteq \mathcal{I}$ and any realization of $y^{N,S}$ in the support of $G_{N,S}(.)$:*
1) The assignment rule is characterized by[32]

$$\left.\begin{array}{r} Q_{0,N,S}(y^{N,S}, m_{\beta,X}^{v-piv}) \\ Q_{j^*,N,S}(y^{N,S}, m_{\beta,X}^{v-piv}) \\ Q^I_{i^*,N,S}(y^{N,S}, m_{\beta,X}^{v-piv}) \end{array}\right\} = 1 \text{ if } \left\{\begin{array}{c} X > \max\{\max_{i\in S}\{x_i^{\text{virt}}(x_i^S)\}, \max_{j\in\{1,\dots,|N|\}}\{x_j^N\}\} \\ x_{j^*}^N > \max\{\max_{i\in S}\{x_i^{\text{virt}}(x_i^S)\}, \max_{j\in\{1,\dots,|N|\}\setminus\{j^*\}}\{x_j^N\}, X\} \\ x_{i^*}^{\text{virt}}(x_{i^*}^S) > \max\{\max_{i\in S\setminus\{i^*\}}\{x_i^{\text{virt}}(x_i^S)\}, \max_{j\in\{1,\dots,|N|\}}\{x_j^N\}, X\} \end{array}\right.$$

2) The payment rule is characterized by the fact that losing bidders do not pay anything and that if the winner is incumbent i [resp. an entrant], he pays $[x_i^{\text{virt}}]^{-1}(SP)$ [resp. SP] where SP denotes the second-highest element in the set $\{x_i^{\text{virt}}(x_i^S), x_j^N, X\}_{\substack{i\in S\\ j=1,\dots,|N|}}$.[33]

We show in the Online Appendix that the virtual pivotal mechanism belongs to the larger class of generalized second-price auctions in which bidding truthfully is a (weakly) dominant strategy.

[32] The cases that are not covered correspond to ties. Any specification will work insofar as ties involving one incumbent occur with probability null (since their virtual valuations contain no atoms) whereas the way the remaining ties are broken do not matter both in terms of buyers' payoffs which are necessarily equal to zero and in terms of the seller's payoff.

[33] If the incumbent i wins the good then he has to pay at least $[x_i^{virt}]^{-1}(X) \geq \underline{x}_i$. Consequently, the incumbent i with valuation $\underline{x}_i$ makes no profit. If we drop the assumption that $x_i^{virt}(\underline{x}_i) \leq X$, then the price rule for incumbent i should be replaced by $[x_i^{virt}]^{-1}(\max\{SP, x_i^{virt}(\underline{x}_i)\})$ and our analysis extend without any difficulty. Note that this also echoes Footnote 27: for incumbents, our virtual pivotal mechanism share the element of Krishna and Perry (1998) that it leaves no rents to the buyers with the lowest type. By contrast, we stress that potential entrants with lowest valuations may make some profit which is a key element in order to match their incentives to enter with their contribution to the virtual welfare

In particular, it is straightforward to check from the payment rule and since the assignment rule guarantees that SP is smaller than the virtual valuation of the winner that the winner never pays more than his valuation. Similarly, since $SP \geq X$ and $[x_i^{virt}]^{-1}(x) \geq x$ for any $i \in \mathcal{I}$, the revenue of the seller is always greater than X. As in Section 4, when $m = m_{\beta,X}^{v\text{-}piv}$ is used, we drop the dependence with respect to $\sigma(m)$ to alleviate notation. We also note that the participation constraints of the incumbents with lowest valuations are binding in the virtual pivotal mechanism, namely

$$V_i^I(\underline{x}_i, m_{\beta,X}^{v\text{-}piv}; q, q^I) = 0 \tag{21}$$

for any $(q, q^I) \in [0,1]^{K+I}$ and any $i \in \mathcal{I}$.

Besides, when A1 holds, the probability of entry must be less than 1 in all groups of entrants in the virtual pivotal mechanism (formally $M(m_{\beta,X}^{v\text{-}piv}) \subseteq [0,1)^K \times [0,1]^I$).[34] As a corollary, we obtain that in equilibrium the seller's revenue coincides with the net virtual welfare in the virtual pivotal mechanism, namely

$$u(\widehat{q}(m_{\beta,X}^{v\text{-}piv}), \widehat{q}^I(m_{\beta,X}^{v\text{-}piv}), m_{\beta,X}^{v\text{-}piv}, X) = NW^{virt}(\widehat{q}(m_{\beta,X}^{v\text{-}piv}), \widehat{q}^I(m_{\beta,X}^{v\text{-}piv}), m_{\beta,X}^{v\text{-}piv}, X). \tag{22}$$

From the point of view of the potential entrants, the virtual pivotal mechanism in our setup with incumbents is equivalent to the efficient second-price auction in a setup without incumbents but in which the valuation of the seller would be stochastically determined according to $\widehat{X}(x^S) := \max\{\max_{i \in S}\{x_i^{virt}(x_i^S)\}, X\}$ after the entry stage where S is the set of incumbents that choose to enter (more formal details are given in the Online Appendix). As shown by Lamy (2013) in Levin and Smith's (1994) model, the fact that the seller's valuation is determined after instead of before entry does not affect the fundamental property of the second-price auction, namely that the payoff of an entrant coincides with his contribution to the net welfare, so that this mechanism induces efficient entry profiles.[35] Here, this congruence property translates into:

$$W_{N_{+k},S}^{v\text{-}piv}(m_{\beta,X}^{v\text{-}piv}, X) - W_{N,S}^{v\text{-}piv}(m_{\beta,X}^{v\text{-}piv}, X) = V_{k,N_{+k},S}(m_{\beta,X}^{v\text{-}piv}) \text{ for any } k \in \mathcal{K}. \tag{23}$$

From (23), simple algebra leads to

$$\frac{\partial NW^{virt}(q, q^I, m_{\beta,X}^{v\text{-}piv}, X)}{\partial q_k} = \mathcal{N}_k \cdot \left[u_k(q, q^I, m_{\beta,X}^{v\text{-}piv}) - C_k\right]. \tag{24}$$

Combining the previous expression with the first-order conditions with respect to the variable

[34] When all the incumbents decide not to enter the virtual pivotal mechanism, then it is equivalent to the efficient second-price auction in which case A1 guarantees that if all potential entrants from a given group k were deciding to enter, then their expected payoff in the auction could not recover their entry cost C_k. If some incumbents were also deciding to enter, then the expected payoff of the entrants decrease which reinforces the argument.

[35] However, this raises some implementation issues since the seller does not have the proper incentives to report truthfully her valuation. Lamy (2013) shows under some conditions that the English auction with cancelation rights, no reserve price and the possibility to submit jump bids implements the first-best. Note also that the results in this paper show that the analysis in Lamy (2013) extends to multiple groups of entrants.

q from the maximization program that characterizes the set $J^{virt}(m^{v\text{-}piv}_{\beta,X}, X)$, we obtain for any $(q, q^I) \in J^{virt}(m^{v\text{-}piv}_{\beta,X}, X)$ that for each $k \in \mathcal{K}$, the potential entrants' equilibrium condition (4) is satisfied for the virtual pivotal mechanism.

By definition, the virtual pivotal mechanism implements the ex post efficient assignment when efficiency is defined according to virtual valuations. We have thus the analog of (2), namely

$$W^{virt}_{N,S}(m, X; \sigma(m)) \leq W^{virt}_{N,S}(m^{v\text{-}piv}_{\beta,X}, X) \tag{25}$$

for any profile of potential entrants N and incumbents S, any mechanism $m \in \mathcal{M}$ and any bidding profile $\sigma(m) \in \Sigma(m)$. As a corollary, we obtain that $NW^{virt}(q, q^I, m, X; \sigma(m)) \leq NW^{virt}(q, q^I, m^{v\text{-}piv}_{\beta,X}, X)$.

If $(\widetilde{q}, \widetilde{q}^I) \in J^{virt}(m^{v\text{-}piv}_{\beta,X}, X)$, it is straightforward that $(\widetilde{q}, \widetilde{q}^I, m^{v\text{-}piv}_{\beta,X})$ joint with truthful bidding is a solution of the maximization program (20). The issue is then whether this virtual first-best can be implemented as an equilibrium in the virtual pivotal mechanism.

More precisely, the problem is that when the winner is an incumbent (say i) then there is no longer a congruence in the virtual pivotal mechanism between his payoff, $x^{\mathcal{I}}_i - [x^{virt}_i]^{-1}(SP)$, and his contribution to the virtual welfare, $x^{virt}_i(x^{\mathcal{I}}_i) - SP$, so that there is no guarantee that incumbents have the proper incentives to enter. Thanks to the regularity assumption, we have nevertheless that the contribution of any incumbent i to the virtual welfare is greater than his payoff (so that if the incumbent is willing to enter for sure, his entry is also surely good from the viewpoint of maximizing the net welfare).[36] Formally we have

$$W^{v\text{-}piv}_{N,S}(m^{v\text{-}piv}_{\beta,X}, X) - W^{v\text{-}piv}_{N,S_{-i}}(m^{v\text{-}piv}_{\beta,X}, X) \geq V_{k,N_{+k},S}(m^{v\text{-}piv}_{\beta,X}) \quad \text{for any} \quad k \in \mathcal{K}. \tag{26}$$

This further implies that for any $i \in \mathcal{I}$,

$$\frac{\partial NW^{virt}(q, q^I, m^{v\text{-}piv}_{\beta,X}, X)}{\partial q^I_i} \geq u^I_i(q, q^I, m^{v\text{-}piv}_{\beta,X}) - \beta^I_i \cdot C_k. \tag{27}$$

For any $(\widetilde{q}, \widetilde{q}^I) \in J^{virt}(m^{v\text{-}piv}_{\beta,X}, X)$, the first-order conditions imply that $\frac{\partial NW^{virt}}{\partial q^I_i}(\widetilde{q}, \widetilde{q}^I, m^{v\text{-}piv}_{\beta,X}, X) \leq 0$ for any incumbent i such that $\widetilde{q}^I_i = 0$. Combined with (27), we obtain finally that $u^I_i(\widetilde{q}, \widetilde{q}^I, m^{v\text{-}piv}_{\beta,X}) \leq \beta^I_i \cdot C^I_i \leq C^I_i$ which guarantees that incumbents that should not participate in the virtual first-best have no incentives to participate in the virtual pivotal mechanism.

Let $\mathcal{Q}^{ex} := \{(q, q^I) \in [0,1]^{K+I} | q^I_i \in \{0, 1\} \quad \text{for any} \quad i \in \mathcal{I}\}$. Since $NW^{virt}(q, q^I, m, X; \sigma(m))$ is linear in q^I, then the restriction $(q, q^I) \in \mathcal{Q}^{ex}$ is without loss of generality when one seeks an element in $J^{virt}(m, X; \sigma(m))$, and in particular for $m = m^{v\text{-}piv}_{\beta,X}$. The next assumption is introduced to guarantee that in the virtual welfare-maximizing entry profile, those incumbents that should enter do recover their entry costs.

[36]The inequality $x^{virt}_i(x^{\mathcal{I}}_i) - SP \geq x^{\mathcal{I}}_i - [x^{virt}_i]^{-1}(SP)$ is equivalent to $\frac{1 - F^I_i(x^{\mathcal{I}}_i)}{f^I_i(x^{\mathcal{I}}_i)} \leq \frac{1 - F^I_i([x^{virt}_i]^{-1}(SP))}{f^I_i([x^{virt}_i]^{-1}(SP))}$, which holds (thanks to A4) when the good is assigned to incumbent i since we have then $x^{virt}_i(x^{\mathcal{I}}_i) \geq SP$.

Assumption A 5 *There exists* $(\widetilde{q},\widetilde{q}^I)\in J^{\text{virt}}(m^{\text{v-piv}}_{\beta,X},X)\cap\mathcal{Q}^{ex}$ *such that for any* $i\in\mathcal{I}$, $\widetilde{q}^I_i=1$ *implies that* $u^I_i(\widetilde{q},\widetilde{q}^I,m^{\text{v-piv}}_{\beta,X})\geq C^I_i$.

Comments: 1) If the supports of the valuation distributions are the same for all buyers, then the rents of the incumbents are necessarily strictly positive in the virtual pivotal mechanism and then A5 holds if the incumbents' entry cost are small enough (while being possibly strictly positive). 2) If $\beta^I_i=0$, then $NW^{virt}(q,q^I,m^{v\text{-}piv}_{\beta,X},X)$ is increasing in q^I_i so that there always exists $(\widetilde{q},\widetilde{q}^I)\in J^{virt}(m^{v\text{-}piv}_{\beta,X},X)$ such that $\widetilde{q}^I=(1,\cdots,1)$. 3) If $\beta^I_i=1$, then both (26) and (27) stand as equalities: This further implies that if $\widetilde{q}^I_i=1$, then we have $\frac{\partial NW^{virt}}{\partial q^I_i}(\widetilde{q},\widetilde{q}^I,m^{v\text{-}piv}_{\beta,X},X)\geq 0$ and then $u^I_i(\widetilde{q},\widetilde{q}^I,m^{v\text{-}piv}_{\beta,X})\geq\beta^I_i\cdot C^I_i$ so that the congruence between private payoffs and welfare is restored (as in Section 3).

4.4 Optimal discrimination with incumbents

Gathering the previous observations about entry incentives in the virtual pivotal mechanism, we obtain that when A5 holds, the corresponding virtual welfare-maximizing entry profile $(\widetilde{q},\widetilde{q}^I)$ is compatible with equilibrium behavior in $m^{v\text{-}piv}_{\beta,X}$. That is, we obtain a (slightly weaker) analog of Lemma 3.1 but in an environment with incumbents:

$$J^{virt}(m^{v\text{-}piv}_{\beta,X},X)\cap M(m^{v\text{-}piv}_{\beta,X})\cap\{[0,1)^K\times\{0,1\}^I\}\neq\emptyset. \tag{28}$$

As a corollary, we obtain that the virtual pivotal mechanism solves the maximization program (20). This implies in turn a partial converse if we make the PG-refinement. This comes after noting that: First, from the perspective of potential entrants the virtual pivotal mechanism is equivalent to the second-price auction with the reserve price set at the maximum of the seller's reservation value and incumbents valuations and is thus a potential game w.r.t. the potential entrants with the total net virtual welfare as the potential function which guarantees that $\widehat{q}(m^{v\text{-}piv}_{\beta,X})\in\operatorname{Arg\,max}_{q\in[0,1]^K}NW^{virt}(q,\widehat{q}^I(m^{v\text{-}piv}_{\beta,X}),m^{v\text{-}piv}_{\beta,X},X)$. Any equilibrium that satisfies the PG-refinement and with $\widehat{q}^I(m^{v\text{-}piv}_{\beta,X})=\widetilde{q}^I$ implements thus the virtual first-best (otherwise posting $m^{v\text{-}piv}_{\beta,X}$ would be a profitable deviation). Second, any equilibrium that implements the virtual first-best is equivalent to the optimal one arising with the virtual pivotal mechanism. To sum up, we obtain the following result:

Proposition 4.1 *Assume that* $m^{\text{v-piv}}_{\beta,X}\in\mathcal{M}$*, A1, A3, A4 and the existence of an entry profile* $(\widetilde{q},\widetilde{q}^I)$ *as defined in A5. There exists an equilibrium in which the seller proposes the virtual pivotal mechanism and the equilibrium entry probability profile is the vector* $(\widetilde{q},\widetilde{q}^I)$*. Such an equilibrium implements the virtual first-best. Conversely, any equilibrium that satisfies the PG-refinement and such that the same set of incumbents as in the above equilibrium enters (formally* $\widehat{q}^I(m^{\text{v-piv}}_{\beta,X})=\widetilde{q}^I$*) is equivalent to such an equilibrium.*

From the perspective of the literature on auctions, Proposition 4.1 can be viewed as providing a general setup in which the optimal design problem can accommodate both exogenous entry (and have thus Myerson (1981) as a special case) and endogenous entry (and have thus our previous non-discrimination result as a special case) while allowing for the mixed case (which the previous literature did not consider and that we think is quite relevant for the study of discrimination).

Proposition 4.1 has several important implications for discrimination. First, it provides an exact characterization of the optimal shape of discrimination. Only incumbents should be discriminated against. That is, even if entrants come from asymmetric group, it is not optimal to treat entrants asymmetrically say in an attempt to reduce the rents left to incumbents. The rents of incumbents should optimally be reduced via the discrimination against incumbents without adding any distortion among entrants. To emphasize the point and also to contrast it with the insights from Myerson (1981), it should be stressed that incumbents should always be discriminated against entrants irrespective of whether incumbents are stronger or weaker than entrants and irrespective of which share of incumbents' profits is internalized by the seller, given that virtual valuations are always below the valuations (namely $x_i^{virt}(x) \leq x$). Furthermore, the shape of discrimination against a given incumbent depends solely on the primitives of this specific incumbent. The information structure with respect to entrants does not play any role and thus does not need to be known to the designer in order to implement the optimal mechanism. Interestingly, as in Myerson (1981), the optimal shape of discrimination does not lead to the complete exclusion of any incumbent, which means that set-asides policies are suboptimal.[37] Nevertheless, if $\widetilde{q}_i^I = 0$ for some $i \in \mathcal{I}$, set-asides can be a useful tool to avoid miscoordination on a suboptimal entry profile. One important difference from a practical perspective is that the implementation of the virtual first-best with the pivotal mechanism does not require the knowledge of the entry costs of the incumbents and the characteristics of potential entrants. By contrast, in order to know which incumbents to exclude (as presumably required with set-asides), much more information is needed.

Further comments:

1) **Dealing with non-regular distributions of incumbents' valuations**

If we drop the assumption that the function $x \to \frac{1-F_i^I(x)}{f_i^I(x)}$ is strictly decreasing (while keeping the assumption that $x-(1-\beta_i^I)\cdot\frac{1-F_i^I(x)}{f_i^I(x)}$ is increasing) then the same conclusion as in Proposition 4.1 would still hold provided the costs C_i^I are small enough to ensure both that incumbent i would enter for sure in equilibrium and that this would be good from the viewpoint of net virtual welfare.

If one of the virtual valuation functions $x_i^{virt}(x)$ is not non-decreasing, then it is impossible to implement the virtual first-best insofar as it would violate the well-known monotonicity constraints, namely that incumbents with higher valuations should receive the good more often. Nevertheless, we can apply Myerson's ironing technique to each of the functions $x-(1-\beta_i^I)\cdot\frac{1-F_i^I(x)}{f_i^I(x)}$, $i \in \mathcal{I}$, and construct a generalized second-price auction with "ironed virtual valuations". From Myerson's (1981) calculations, this solution maximizes the expected virtual welfare $\sum_{N\in\mathbb{N}^K}\sum_{S\subseteq\mathcal{I}} P(N|q)\cdot P(S|q^I)\cdot W_{N,S}^{virt}(m,X;\sigma(m))$ for any $(q,q^I)\in[0,1]^{K+I}$ and any strategy

[37] Indeed, it may occur that the virtual first-best leads to the exclusion of some incumbents (as also some entrants) but in this case there will be an equilibrium in the virtual pivotal mechanism where they prefer not to enter.

$\sigma(m)$ which is compatible with equilibrium behavior at the bidding stage. Maximizing the virtual welfare with respect (q, q^I) provides then an upperbound for the seller's objective. To conclude that this candidate solution is optimal, it is enough to realize that it still gives the right incentives in terms of participation rates as developed in the Online Appendix.

2) **What if the seller can approach the incumbents before the entry decision?**

To the extent that incumbents are ex ante identified by the seller, it may be argued that, if law permits, the seller could approach the incumbents before the start of the mechanism. In the informational scenario considered above, incumbents had no private information from the start and thus approaching them ex ante would allow to restore the first-best outcome without the use of discriminatory devices through the use of well chosen incumbent-specific fees. But, consider an alternative scenario in which the incumbents would privately know their valuations and entry costs from the start. The best mechanism would then take a form identical to the one shown in Proposition 4.2 in which one should deduct the entry cost from the valuation and assume incumbents never pay the entry cost unless they win the object (here, we have in mind that the incumbents' entry costs may be avoided until a decision is reached about who the winner of the good is).

3) **What if discrimination takes the form of a linear distortion of bids?**

The optimal discrimination as arising from the shape of the virtual valuations need not be implementable using standard auctions, say second or first-price auctions, in which the submitted bids would be linearly transformed before the rule of the auction is applied. If one applies such additional constraints on the shape of discrimination, one may be interested in the shape of the optimal slopes that should be applied to the distortion of bids. In the Online Appendix, we consider a simple environment with a single group of potential entrants in the Poisson model discussed in Section 5, and in line with Proposition 4.1 above we show that incumbents should be discriminated against entrants even in this restricted class of discriminatory mechanisms.

4) **Split-awards**

Alternative non-discriminatory instruments sometimes used in practice that allows to discriminate indirectly between bidders are split awards: Instead of assigning a contract entirely to a single firm, it consists in splitting the contract among several of the bidders (instead of just the bidder with highest bid). For example, the bidders submitting the two highest bids split the award in some proportion. One intuitive appeal of such a mechanism in contexts with a pre-auction investment stage is that guaranteing a share of the project to a bidder with non-maximal bid may give weaker bidders a stronger incentive to invest ex ante thereby inducing a more balanced competition ex post and higher revenues (Anton and Yao (1989) and Gong, Li and McAfee 2011).

In the context of our model, assuming that the good is divisible and that valuations are linear in quantity, we obtain as a by-product of Proposition 4.1 that split awards are necessarily suboptimal once the set of possible mechanisms includes the virtual pivotal mechanism. Indeed in an optimal mechanism, the good should be put entirely into the hands of the buyer with the highest virtual valuation (which is generically unique) as in Myerson (1981).

Assuming that the seller cannot use the virtual pivotal mechanism, it would be of interest to analyze whether the use of split awards could increase revenues. Intuitively, split awards seem to be a way to reduce incumbents' rents if incumbents have typically larger valuations and these could thus be desirable from a revenue viewpoint. Yet, split awards also have the drawback of reducing ex post welfare, which is not desirable. Trading-off these two effects from a more positive analysis perspective would require further work.

5 When the numbers of potential entrants grow large

In this Section, we assume that there are no incumbents and that the seller seeks to maximize revenues. An important limitation of our non-discrimination result is that it does not rule out the possibility of equilibria in which the seller would use mechanisms other than the efficient second-price auction. As already mentioned, when the seller posts the efficient second-price auction, it may be the case that equilibria other than the efficient one arises at the entry stage, thereby leaving the door open to the emergence of equilibria which do not implement the first-best and that possibly involve some form of discrimination.

Another concern one may have about our non-discrimination result is that it requires that the seller can use second-price auctions, but in applications like procurement auctions, first-price auctions are much more prevalent. A question arises as to whether first-price auctions perform as well as second-price auctions (especially in the presence of entrants coming from different groups).

The main insights developed below are that when the number of potential entrants in each group is very large, the above two concerns have no bite. First, in the limit as the number of potential entrants grows large in each group, all equilibria are equivalent to the one arising in the no-discrimination equilibrium of Proposition 3.2. Second, first-price auctions become equivalent to second-price auctions (which stands in sharp contrast of the analysis of auctions with asymmetric bidders and exogenous participation).

To formalize this, we work in the limit model in which the number of potential entrants per group is infinite, which together with our assumption that entry decisions are made symmetrically among buyers of the same group leads us to assume that the effective number of entrants from a given group $k \in \mathcal{K}$ of potential entrants is taken to be the realization of a random variable following a Poisson distribution with mean $\mu_k \geq 0$. That is, the probability that there are n_k entrants from each group $k \in \mathcal{K}$ is equal to $e^{-\sum_{k=1}^{K} \mu_k} \cdot \prod_{k=1}^{K} \frac{[\mu_k]^{n_k}}{n_k!}$. The Poisson distribution corresponds to the limit distribution of the number of entrants of each group of a model with a finite number of buyers per group and as the number of buyers in each group goes to infinity (and assuming every individual entrant of a given group follows the same participation strategy).[38] In the Online Appendix, we give more formal details about our equilibrium analysis with Poisson distributions and its foundation as the limit of the equilibrium concept proposed in Definition

[38] By contrast with the voting literature with Poisson games initiated by Myerson (1998, 2002) where the Poisson distributions are taken as exogenous, the parameters μ_k, $k \in \mathcal{K}$, are endogenously determined in our competitive equilibrium.

1 with a finite number of potential entrants.[39] Our previous analysis can be adapted easily by replacing $q \in [0,1]^K$ (resp. $\widehat{q}(m) \in [0,1]^K$) with $\mu \in R_+^K$ (resp. $\widehat{\mu}(m) \in R_+^K$) and letting

$$P(N|\mu) = P_k(N|\mu) = e^{-\sum_{k'=1}^{K} \mu_{k'}} \cdot \prod_{k'=1}^{K} \frac{[\mu_{k'}]^{n_{k'}}}{n_{k'}!}. \tag{29}$$

The property that $P(N|\mu) = P_k(N|\mu)$ for any $k \in \mathcal{K}$ -which is referred to as environmental equivalence in Myerson (1998)- plays a key role in the sequel.

Comment: It should be noted that even if the number of observed participants is not very large in a number of applications, our large market assumption concerns the number of potential entrants, which can be up to four times larger in many cases (see Athey et al. (2011, 2013)).

5.1 Equilibrium uniqueness

When the efficient second-price auction is proposed, then thanks to (11) we obtain after similar calculations to the ones made in the binomial model that

$$\frac{\partial NW(\mu, m_X^{ESP}, X)}{\partial \mu_k} = \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot V_{k,N_{+k}}(m_X^{ESP}) - C_k = u_k(\mu, m_X^{ESP}) - C_k \tag{30}$$

where $NW(\mu, m, X; \sigma(m)) = \sum_{N \in \mathbb{N}^K} P(N|\mu) W_N(m, X; \sigma(m)) - \sum_{k=1}^{K} \mu_k \cdot C_k$. In the efficient second-price auction, the profile of entry rates that are compatible with equilibrium behavior can thus be expressed as:

$$M(m_X^{ESP}) = \{\mu \in R_+^K | \frac{\partial NW}{\partial \mu_k}(\mu, m_X^{ESP}, X) \underset{\text{resp. } \leq}{=} 0 \text{ if } \mu_k \underset{\text{resp. } =}{>} 0 \text{ for each } k \in \mathcal{K}\}. \tag{31}$$

In the binomial model, we have established that $J(m_X^{ESP}, X) \subseteq M(m_X^{ESP})$. In the Poisson model, we have a much stronger result:

$$M(m_X^{ESP}) = \text{Arg} \max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X) \equiv J(m_X^{ESP}, X). \tag{32}$$

This comes from the fact that the function $\mu \to NW(\mu, m_X^{ESP}, X)$ is globally concave.

Lemma 5.1 *For any $X \in \mathbb{R}_+$, $\mu \to NW(\mu, m_X^{\text{ESP}}, X)$ is concave on R_+^K.*

As a corollary, we obtain that in equilibrium the seller's revenue is necessarily equal to the upperbound $\max_{\mu \in R_+^K, m \in \mathcal{M}, \sigma(m) \in \Sigma(m)} NW(\mu, m, X; \sigma(m))$ or equivalently that any equilibrium implements the first-best, because otherwise the seller would strictly gain by proposing the efficient second-price auction that guarantees the seller can reach this bound.

[39] We drop A1 in the Poisson model. This corresponds to the observation that we cannot have $\mu_k = +\infty$ in equilibrium.

Proposition 5.2 *Consider the Poisson model in an environment without incumbents, assume that $m_X^{\text{ESP}} \in \mathcal{M}$ and that the seller seeks to maximize revenues. Any equilibrium is equivalent to an equilibrium in which the seller proposes the efficient second-price auction and the first-best is achieved.*

Interestingly, Roberts and Sweeting (2012) notes informally that the equilibrium multiplicity issue seems to disappear (in simulations) once the number of potential entrants gets large in each group. Here, we formalize this insight for second-price auctions.[40]

To the best of our knowledge, no such "uniqueness" result appears in the earlier auction literature with endogenous participation at least in a model of such generality. In this respect, we stress that our result does not hinge on restrictions about the distributions of valuations or on the entry costs. As it turns out, the assumption that the set of potential entrants is large is key for the derivation of Proposition 5.2. To illustrate why it would not hold when the set of potential entrants is finite, consider the following simple scenario. There are two types of entrants $K = 2$ with $F_1(x) = \mathbf{1}[x \geq 1]$, $F_2(x) = \mathbf{1}[x \geq 1+\epsilon]$, $\epsilon > 0$, $C_1 = C_2 = C \in (\epsilon, 1)$ and $X = 0$. Assume first that there is only one potential buyer per group. There are then three equilibria in the second-price auction: the two pure strategy equilibria in which one buyer enters for sure and not the other one, and a purely mixed equilibrium in which buyer 1 [resp. 2] enters with probability $q_1 = 1 + \epsilon - C$ [resp. $q_2 = 1 - C$]. Only the equilibrium in which buyer 2 enters for sure maximizes the net welfare. Putting several potential buyers per group has a concavification effect on how net welfare depends on the entry rates, which in turn guarantees uniqueness.

To illustrate why increasing the number of potential entrants alleviates the miscoordination problem between bidders from different groups, we develop further the previous toy example by considering multiple potential buyers per groups. Specifically, suppose now that there are $\mathcal{N}$ buyers instead of one buyer in group 1, while we still assume for simplicity that there is just one potential buyer in group 2. There are three candidate equilibria. First the one in which only the buyers from group 1 are active ($q_1 > 0$ and $q_2 = 0$), second the one in which only the buyer from group 2 is active ($q_1 = 0$ and $q_2 = 1$) and finally the one in which buyers from both groups are active ($q_1, q_2 \notin \{0, 1\}$). The second candidate equilibrium implements the efficient entry profile and is always an equilibrium. By contrast, the two other candidate equilibria involve inefficiencies and may not be equilibria. The first one requires that $(1 - q_1)^{\mathcal{N}-1} = C$ to make buyers from group 1 indifferent between entering and not. It is an equilibrium if an only if the buyer from group 2 does not find it profitable to enter, that is, whenever $\epsilon \leq C - C^{\frac{\mathcal{N}}{\mathcal{N}-1}}$. The last candidate equilibrium imposes (to make buyers 1 and 2 indifferent) that $(1-q_1)^{\mathcal{N}-1} \cdot (1-q_2) = C$ and $(1-q_1)^{\mathcal{N}} = C - \epsilon$, which again also requires that $\epsilon \leq C - C^{\frac{\mathcal{N}}{\mathcal{N}-1}}$. To sum up, an inefficient equilibrium can arise only when $\epsilon \leq C - C^{\frac{\mathcal{N}}{\mathcal{N}-1}}$, a condition that cannot hold when $\mathcal{N}$ is large enough.[41]

[40] Equilibrium multiplicity calls for a selection rule. In the empirical literature, it typically consists in picking the welfare-maximizing equilibrium. Our work gives some theoretical support for this selection rule (at least in second-price auctions) insofar as this equilibrium survives when the number of bidders per group gets large.

[41] For any finite $\mathcal{N}$, the previous example shows that we can cook a setup such that inefficiencies may occur by taking ϵ small enough. But, for any fixed $\epsilon > 0$, no such equilibrium exists if $\mathcal{N}$ is large enough.

More generally, when the number of potential entrants is large in each group and group k characteristics unambiguously dominate group k' characteristics (both in terms of having a lower entry cost and a distribution of valuation that first-order stochastically dominates that of group k'), it is not possible in equilibrium that buyers of group k' would enter with positive probability as otherwise all buyers of group k would find it profitable to enter, which is not feasible.[42]

Comment: In the presence of incumbents, there is no hope in general that the function $(\mu, q^I) \to NW^{virt}(\mu, q^I, m, X)$ be globally concave for any required mechanism m, and in particular for $m = m^{v\text{-}piv}_{\beta,X}$. Nevertheless, we can show that the function $\mu \to NW^{virt}(\mu, q^I, m^{v\text{-}piv}_{\beta,X}, X)$ is concave for any vector $q^I \in [0,1]^I$ since it can be viewed as a convex combination of welfare functions of the type considered in Lemma 5.1 (more specifically, we have $NW^{virt}(\mu, q^I, m^{v\text{-}piv}_{\beta,X}, X) = \sum_{S \subseteq \mathcal{I}} P(S|q^I) \cdot E_{x^S}[NW^{virt}(\mu, m^{v\text{-}piv}_{\beta,\widehat{X}(x^S)}, \widehat{X}(x^S))]$ where $\widehat{X}(x^S) := \max\{\max_{i \in S}\{x_i^{virt}(x_i^{\mathcal{I}})\}, X\}$).[43] This means that for a given set of incumbents, then the virtual pivotal mechanism must induce entry rates that maximize the net virtual welfare. In particular, if the entry costs of the incumbents are small enough so that incumbents enter with probability 1, then any equilibrium is equivalent to the one exhibited in Proposition 4.1 and thus any equilibrium implements the virtual first-best.

5.2 First-price auctions

We maintain our assumptions that there are no incumbents and that the number of potential entrants is infinite in each group. We show that the outcome of the efficient second-price auction can also be implemented using a first-price auction in which the reserve price is set at the seller's valuation, which we believe has some applied value given that, in most procurement auctions, the auction format is of that form. In the rest of the section, we refer to this format as the efficient first-price auction.

Throughout this section, we also make the following additional assumption on the information structure:

Assumption A 6 *Buyers do not receive any information in addition to their private valuation. The distributions $F_k(.|z)$, $k \in \mathcal{K}$, do not depend on z and are continuously differentiable on their (common) support which is denoted by $[\underline{x}, \overline{x}]$.*

In particular, valuations are now assumed to be drawn independently. Next, we drop the dependence in z in the notation. We stress that A6 assumes implicitly that entrants do not receive any information about the other entrants, in particular the number they are, at the moment they submit their bid. Note however that we impose no symmetry assumption on entrants given that the distribution F_k (as well as the entry costs C_k) may vary with the group k.

[42] This is so because both kinds of bidders expect to be facing approximately the same distribution of opponents from group k'. Having one more or one less potential buyer does not change much the binomial distribution which is close to the Poisson distribution.

[43] More formal details are given in the Online Appendix.

An entrant from group k expects that the probability that the realization of the profile of opponents he faces is N is given by $P_k(N|\mu)$. From the so-called "environmental equivalence" property that arises with Poisson distributions, this probability coincides with $P(N|\mu)$ and thus does not depend on k. This implies that if all buyers are using the same strictly increasing bidding function $B : R_+ \to R_+$ with $B(r) = r$ in the first-price auction with reserve price r, then each entrant independently of the group he comes from expects to be facing the same distribution of bids of other participants. More precisely, the best-response of an entrant with valuation $x \geq r$ is to submit an active bid and he expects then to get[44]

$$u^{FP}(x;r) = \max_{x' \in [r,\overline{x}]} \left\{ (x - B(x')) \cdot \prod_{k=1}^{K} e^{-\mu_k(1-F_k(x'))} \right\} \tag{33}$$

no matter which group he comes from. Because this maximization program is independent of the group k, one can then ensure that the bidding strategy of participants is independent of the group (and depends solely on the valuation). More precisely, we obtain from standard arguments in auction theory (see Krishna, 2002), that for the first-price auction with reserve price r and for any entry profile μ, there exists a symmetric equilibrium at the bidding stage in which every bidder with valuation x bids according to $B(x) = \int_{\underline{x}}^{x} \max\{y, r\} \frac{d[\prod_{k=1}^{K} e^{-\mu_k(1-F_k(y))}]}{\prod_{k=1}^{K} e^{-\mu_k(1-F_k(x))}}$ if $x \geq r$ and $B(x) < r$ (or equivalently non-participation) otherwise. In equilibrium, buyers are bidding the expectation of the highest valuation among their opponents (interpreting the reserve price as the valuation of the seller) conditional on having the highest valuation. As a consequence of this, one may sustain an equilibrium in which the seller uses the efficient first-price auction, entry rates are determined according to the first-best values and buyers get the same expected utility as in the efficient second-price auction.[45] Formally,

Proposition 5.3 *Consider the Poisson model in an environment without incumbents, assume that the efficient first-price auction belongs to $\mathcal{M}$, that the sellers seeks to maximize revenues and that A6 hold. There exists an equilibrium in which the seller proposes the efficient first-price auction and it implements the first-best.*

Proposition 5.3 seems to be inconsistent with Athey et al.'s (2011) structural estimates which have a special focus on the non-equivalence between first- and second-price auctions. One of the differences comes from the fact that we assume implicitly in A6 that buyers do not know the set of participants at the bidding stage. By contrast, Athey et al. (2011) -as most of the empirical literature- assumes that the set of participants is common knowledge among bidders so that the bidding stage is the same as in models with exogenous entry in which stronger buyers are bidding less aggressively (Maskin and Riley, 2000). To the best of our knowledge, there is no clear-cut

[44]From A6, the CDFs $F_k(.)$ have no atoms so that ties would occur with a null probability and we can thus abstract from the possibility of ties.

[45]The fact that the payoffs are the same is a consequence of the celebrated payoff equivalence result in private value auctions stating that if two mechanisms allocate the good in the same way -and non-winning agents get 0 utility- the expected payment made by buyers must be the same conditional on their valuation.

empirical justification for one assumption or another. Under the assumption that bidders do not know the set of participants, we obtain the insight that the way bidders shade their bids become homogenous in the limit when the number of potential entrants goes to infinity. Another more normative lesson that comes out of this is that it may be preferable to not let the bidders know the profile of other participants at the time they must submit their bids.

Comment: The above Proposition does not establish the uniqueness insight obtained in the previous Subsection. Yet, we still get uniqueness if we assume that the symmetric equilibrium exhibited above at the bidding stage is always played.[46]

6 Extensions

6.1 The seller is privately informed of her valuation

Consider the setting of Section 3. In some applications, it makes sense to assume that only the seller knows her valuation X, where X is drawn from some arbitrary distribution. A priori, we move now into the territory of informed principal problems in which the choice of format may convey some information to the buyers. This signaling aspect is often a source of multiplicity in principal-agent settings. Yet, as in Myerson (1981), this is not so in our context. Indeed, by choosing the efficient second-price auction (given her valuation), a seller whatever her valuation generates the highest possible expected net welfare and her expected revenue is equal to this welfare (net of the participation costs of the entrants). Suppose now that the seller were to pick the same auction format for a pool of different valuations and that such a mechanism were to generate some positive probability of entry. Given that first-best entry probabilities if positive cannot be the same with different reservation values, we obtain then that the net welfare would be strictly lower than the one arising in the situation in which the seller for all these valuations would pick the efficient second-price auction. However, for this pool of valuations, the welfare and the revenue should coincide on average as entrants' expected payoffs should coincide with their entry cost. This further implies that the seller with at least one pooled valuation would be strictly better off choosing the efficient second-price auction, thereby leading to a contradiction. Thus,[47,48]

Proposition 6.1 *Assume that $\mathcal{M}^r_{SP} \subseteq \mathcal{M}$, and that A1 and A2 hold. Consider an equilibrium with an informed seller such that $(q(m_X^{ESP}), q^I(m_X^{ESP})) \in J(m_X^{ESP}, X)$ for any X. On the equilibrium path, a mechanism that attracts some entrants cannot be proposed by sellers with different valuations. For any realization of the type of the seller, the strategy profile on the equilibrium path*

[46] To the best of our knowledge, equilibrium uniqueness results have not been established in first-price auctions with a stochastic number of bidders. We conjecture that this is the unique (group-symmetric) equilibrium.

[47] The argument as to why assuming the valuation X is private information to the seller makes no difference is somewhat related to some insights appearing in the literature on informed principals (see e.g. Maskin and Tirole, 1990). Here, we are in a private value setup (i.e., the private information of the seller does not directly affect buyers' preferences). Moreover, from an ex ante perspective the seller can not do better than in the situation in which her private information would be known to the buyers, which thereby is suggestive why the seller has no interest in not disclosing her information. Despite these general observations, we cannot rely on the existing results of the informed principal literature because here participation is endogenous unlike in this literature.

[48] In Appendix, we provide precisions on the definition of an equilibrium in this environment.

is equivalent to the one in which the seller always proposes the efficient second-price auction. Such an equilibrium implements the first-best.

In the Poisson model without incumbents, we have that the analog of the extra assumption $q(m_X^{ESP}) \in J(m_X^{ESP}, X)$ holds automatically and thus we get a uniqueness insight:

Corollary 6.2 *In the Poisson model without incumbents, any equilibrium with an informed revenue-maximizing seller is equivalent to an equilibrium where she posts efficient second-price auctions and it implements the first-best.*

6.2 Multi-object auctions

Our existence result of an equilibrium that implements the first-best extends straightforwardly to any multi-object assignment problem with private values provided one replaces the efficient second-price auction with the pivotal mechanism which is the Groves mechanism in which agents pay the social surplus loss their presence inflicts on others (see Jehiel and Lamy (2015) for elaborations on this in the context of provision of public goods). What is less clear is whether the uniqueness insight holds in multi-object assignment problems. Under some specific structures (in particular when a bundle is valued by all buyers according to the sum of its individual values), one can show that in the Poisson model the efficient second-price auction on each object still induces for sure an entry profile that maximizes the net welfare so that Proposition 5.2 carries over as detailed in the Online Appendix. From a technical perspective, one can establish in this case that the net welfare function is globally concave as a function of the entry rates. One obtains thus as a corollary that bundling would be detrimental to the seller, an insight which contrasts with the multi-object literature with exogenous entry in which the optimal mechanism involves a departure from full efficiency (Jehiel and Moldovanu 2001) and some form of bundling (Jehiel et al. 2007) even in the additive case and when valuations are drawn independently across objects.

Another interesting multi-object application that we are able to cover is the sponsored search auction setup à la Edelman et al. (2007) and Varian (2007). These authors consider the following model. There are L units of (possibly) different sizes taken from an homogenous good and bidders are allowed to win at most one unit. The size of the k^{th} unit is denoted by s_k and units are ordered so that $s_1 \geq \cdots \geq s_L$. Each buyer is characterized by a valuation x so that his valuation for the k^{th} unit is given by $s_k \cdot x$ for any k. In this environment, the pivotal mechanism consists in assigning the k^{th} unit to the buyer with the k^{th} highest valuation (or bid) which is denoted by p_k and making the latter pay $s_k \cdot p_{k+1} - \sum_{i=k+1}^{L} s_i \cdot (p_i - p_{i+1})$. Again one can establish in the pivotal mechanism that the net welfare function is globally concave as a function of the entry rates. The efficiency of the optimal auction with endogenous participation contrasts with the optimal auction with exogenous participation characterized by Edelman and Schwarz (2010).

7 Conclusion

Our main insight is that considering that participation is endogenously determined by the choice of the auction format deeply affects how one should think of discrimination in procurement auctions. There should be no discrimination among potential entrants whose rents are inelastic to the choice of mechanism. By contrast, when there are incumbents whose participation decisions are independent of the mechanism, those should be discriminated against entrants no matter whether they are ex ante stronger or weaker than entrants and no matter which share of their surplus is internalized by the designer. As the number of potential entrants grows large in each group (or alternatively under the refinement that buyers coordinate on the equilibria that maximize the potential function in potential games), any equilibrium is equivalent to the equilibrium supporting these insights.

We should be a bit cautious about how to interpret our insights from a policy perspective and in particular in procurements where competition is not only in price, since quality is also often used as an award criterion. Since quality is subjective and since the objective of the evaluators of those subjective criteria may not stand in line with those of the designer, then parts of the procurement can be intrinsically subject to some form of discrimination though the rules are explicitly non-discriminatory. The question is then whether the designer should counterbalance those informal discriminatory practices w.r.t. quality by using explicit discriminatory practices w.r.t. to prices. The answer depends on whether the informal discrimination takes place ex ante or ex post.

On the one hand, if implicit discrimination takes place at the ex post stage, e.g. as the quality scores of some bidders are inflated, then explicit discrimination could be a useful tool to restore equal treatment by counterbalancing the unequal treatment in the way bids are assessed.[49] Indeed the first-best would be attained if the policymaker could impose a scoring rule that restores a proper evaluation of the quality. However, if this is not practically feasible, natural instruments include bid preferences as a second-best.

On the other hand, if implicit discrimination takes place at the ex ante stage, e.g. as quality requirements in the contract are specified in a way that is advantageous to some bidders, then counterbalancing this distortion by penalizing those bidders at the bidding stage may be detrimental: In particular, once the terms of contracts are specified, then Proposition 3.2 is still at work and pleads in favor of non-discrimination. Instead of canceling out each other, further distortions would be detrimental to the designer. This logic relies on the assumption that the ex ante advantaged bidders form a group of potential entrants. By contrast, if ex ante privileged bidders participate for sure (due to their privilege), in such a case such bidders should be viewed as incumbents in our framework, and the conclusion of Proposition 4.1 suggests that ex ante privileged bidders should be discriminated against at the bidding stage but in a way that depends on the distribution of their type (which shapes their informational rents) rather than on the initial distortions themselves.

[49]Enhancing transparency is also a way to limit such distortions but only to some extent.

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Appendix

Proof of Lemma 3.1

As a preliminary, we give some useful formulas for the efficient second-price auction. Using standard results from auction theory, conditional on z, a buyer with valuation $u \geq X$ who participates in the seller's auction against the profile $N \in \mathbb{N}^K$ of entrants and S for the incumbents when the reserve price is X will receive the expected payoff of $\int_X^u \prod_{k=1}^K [F_k(x|z)]^{n_k} \cdot \prod_{i\in S} F_i^I(x|z)dx$.[50] The corresponding (interim) payoff of a group k buyer from entering such an auction, i.e. before knowing what his valuation will be, is given (after simple calculations) by

$$V_{k,N_{+k},S}(m_X^{ESP}) = E_Z\Big[\int_X^\infty \prod_{k'=1}^K [F_{k'}(x|z)]^{n_{k'}} \cdot \prod_{i\in S} F_i^I(x|z)\cdot(1-F_k(x|z))dx\Big] = \int_X^\infty (F^{(1:N\cup S)}(x) - F^{(1:N_{+k}\cup S)}(x))dx. \tag{34}$$

Similarly we have $V_{i,N,S}^I(m_X^{ESP}) = \int_X^\infty (F^{(1:N\cup S_{-i})}(x) - F^{(1:N\cup S)}(x))dx$ for any incumbent i. From (34), it is straightforward that

$$V_{k,N_{+k},S}(m_X^{ESP}) \leq E_Z\Big[\int_X^\infty [F_k(x|z)]^{n_k}(1-F_k(x|z))dx\Big]. \tag{35}$$

From Assumption A1, we obtain then that for any $k \in \mathcal{K}$, we have $V_{k,N,S}(m_X^{ESP}) < C_k$ if $n_k = \mathcal{N}_k$, which further implies by taking expectations that

$$q_k = 1 \Rightarrow u_k(q, q^I, m_X^{ESP}) < C_k. \tag{36}$$

From the equilibrium condition (4), we obtain that $(q, q^I) \in M(m_X^{ESP})$ implies that $q_k < 1$, or equivalently $M(m_X^{ESP}) \subseteq [0,1)^K \times [0,1]^I$.

The expression of the expected welfare is given by

$$W_{N,S}(m_X^{ESP}, X) = X \cdot F^{(1:N\cup S)}(X) + \int_X^\infty x d[F^{(1:N\cup S)}(x)] = X + \int_X^\infty (1 - F^{(1:N\cup S)}(x))dx \tag{37}$$

where the last equality comes after an integration per part. From (34) and (37), we obtain

$$W_{N_{+k},S}(m_X^{ESP}, X) - W_{N,S}(m_X^{ESP}, X) = V_{k,N_{+k},S}(m_X^{ESP}) \tag{38}$$

for any $k \in \mathcal{K}$ and similarly, for any $i \in S$

$$W_{N,S}(m_X^{ESP}, X) - W_{N,S_{-i}}(m_X^{ESP}, X) = V_{i,N,S}^I(m_X^{ESP}). \tag{39}$$

[50] This is the integral of the (interim) probability that a bidder with valuation x wins the object as x varies from X to u conditional on z.

In words, we have thus that the social contribution of any kind of buyer (either a potential entrant or an incumbent) coincides with his contribution to the welfare, which is the fundamental property of the efficient second-price auction.

Comment: This congruence holds indeed for any realization of the valuations and thus a fortiori on average as captured in (38) and (39).

We now establish Lemma 3.1 formally. We note as a preliminary that $\frac{\partial P(N|q)}{\partial q_k} = -\mathcal{N}_k \cdot P_k(N|q)$ if $n_k = 0$, $\frac{\partial P(N|q)}{\partial q_k} = \mathcal{N}_k \cdot P_k(N_{-k}|q)$ if $n_k = \mathcal{N}_k$ and $\frac{\partial P(N|q)}{\partial q_k} = -\mathcal{N}_k \cdot P_k(N|q) + \mathcal{N}_k \cdot P_k(N_{-k}|q)$ if $n_k \in [1, \mathcal{N}_k - 1]$. For any $m \in \mathcal{M}$ and $k \in \mathcal{K}$, we have then

$$\begin{aligned}
\frac{\partial NW(q,q^I,m,X;\sigma(m))}{\partial q_k} &= \sum_{N\in\prod_{k'=1}^{K}[0,\mathcal{N}_{k'}]} \sum_{S\subseteq\mathcal{I}} \frac{\partial P(N|q)}{\partial q_k} \cdot P(S|q^I) \cdot W_{N,S}(m,X;\sigma(m)) - \mathcal{N}_k \cdot C_k \\
&= -\mathcal{N}_k \cdot \sum_{N\in\prod_{k'=1}^{k-1}[0,\mathcal{N}_{k'}]\times[0,\mathcal{N}_k-1]\times\prod_{k'=k+1}^{K}[0,\mathcal{N}_{k'}]} \sum_{S\subseteq\mathcal{I}} P_k(N|q) \cdot P(S|q^I) \cdot W_{N,S}(m,X;\sigma(m)) \\
&+ \mathcal{N}_k \cdot \sum_{N\in\prod_{k'=1}^{k-1}[0,\mathcal{N}_{k'}]\times[1,\mathcal{N}_k]\times\prod_{k'=k+1}^{K}[0,\mathcal{N}_{k'}]} \sum_{S\subseteq\mathcal{I}} P_k(N_{-k}|q) \cdot P(S|q^I) \cdot W_{N,S}(m,X;\sigma(m)) - \mathcal{N}_k \cdot C_k \\
&= \mathcal{N}_k \cdot \left(\sum_{N\in\mathbb{N}^K} \sum_{S\subseteq\mathcal{I}} P_k(N|q) \cdot P(S|q^I) \cdot [W_{N_{+k},S}(m,X;\sigma(m)) - W_{N,S}(m,X;\sigma(m))] - C_k \right).
\end{aligned} \tag{40}$$

For $m = m_X^{ESP}$, we obtain then from (38) that

$$\frac{\partial NW(q,q^I,m_X^{ESP},X)}{\partial q_k} = \mathcal{N}_k \cdot [u_k(q,q^I,m_X^{ESP}) - C_k]. \tag{41}$$

For incumbents, we have similarly: for any $S \subseteq \mathcal{I}$ and $i \in S$, $\frac{\partial P(S|q^I)}{\partial q_i^I} = -\frac{\partial P(S_{-i}|q^I)}{\partial q_i^I} = \prod_{i'\in S_{-i}} q_{i'}^I \cdot \prod_{i'\in\mathcal{I}\setminus S}(1-q_{i'}^I)$. For any $m \in \mathcal{M}$ and $i \in \mathcal{I}$, we have then

$$\begin{aligned}
\frac{\partial NW(q,q^I,m,X;\sigma(m))}{\partial q_i^I} &= \sum_{N\in\mathbb{N}^K} \sum_{S\subseteq\mathcal{I}} P(N|q) \cdot \frac{\partial P(S|q^I)}{\partial q_i^I} \cdot W_{N,S}(m,X;\sigma(m)) \\
&= \sum_{N\in\mathbb{N}^K} \sum_{\substack{S\subseteq\mathcal{I}\\ i\in S}} P(N|q) \cdot \Big[\frac{\partial P(S|q^I)}{\partial q_i^I} \cdot W_{N,S}(m,X;\sigma(m)) + \frac{\partial P(S_{-i}|q^I)}{\partial q_i^I} \cdot W_{N,S_{-i}}(m,X;\sigma(m))\Big] - C_i^I \\
&= \sum_{N\in\mathbb{N}^K} \sum_{\substack{S\subseteq\mathcal{I}\\ i\in S}} P(N|q) \cdot \prod_{i'\in S_{-i}} q_{i'}^I \cdot \prod_{i'\in\mathcal{I}\setminus S}(1-q_{i'}^I) \cdot \Big[W_{N,S}(m,X;\sigma(m)) - W_{N,S_{-i}}(m,X;\sigma(m))\Big] - C_i^I.
\end{aligned} \tag{42}$$

For $m = m_X^{ESP}$, we obtain then from (39) that

$$\frac{\partial NW(q,q^I,m_X^{ESP},X)}{\partial q_i^I} = u_i^I(q,q^I,m_X^{ESP}) - C_i^I. \tag{43}$$

Consider $(q,q^I) \in J(m,X;\sigma(m))$. The first-order conditions for the optimality of (q,q^I) lead to:

$$\frac{\partial NW}{\partial q_k}(q,q^I,m,X;\sigma(m)) \underset{resp. \binom{\leq}{\geq}}{=} 0 \text{ if } q_k \underset{resp.=\binom{0}{1}}{\in} (0,1) \tag{44}$$

for each $k \in \mathcal{K}$ and $\frac{\partial NW}{\partial q_i^I}(q, q^I, m, X; \sigma(m)) \underset{resp.\binom{\leq}{\geq}}{=} 0$ if $q_i^I \underset{resp.=\binom{0}{1}}{\in (0,1)}$ for each $i \in \mathcal{I}$.

Combining (41) and (44) for potential entrants and the analog properties for incumbents, we obtain that any pair $(q, q^I) \in J(m_X^{ESP}, X)$ is compatible with equilibrium behavior, namely $(q, q^I) \in M(m_X^{ESP})$, which concludes the proof.

Proof of Proposition 3.3

For a given $(\widetilde{q}, \widetilde{q}^I) \in J(m_X^{ESP}, X)$, we let $\widehat{m} = m_X^{ESP}$, $(\widehat{q}(m), \widehat{q}^I(m)) = (\widetilde{q}, \widetilde{q}^I)$ and $\widehat{\sigma}(m)$ be the truthful strategy if $m = m_X^{ESP}$ and we pick any undominated strategy $\widehat{\sigma}(m) \in \Sigma(m)$ and any participation rates $(\widehat{q}(m), \widehat{q}^I(m)) \in M(m; \sigma(m))$ for $m \in \mathcal{M} \setminus \{m_X^{ESP}\}$ (which is possible since $M(m; \sigma(m)) \neq \emptyset$ for any m and $\sigma(m)$). From Lemma 3.1, we obtain that $(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m})) \in M(\widehat{m}; \widehat{\sigma}(\widehat{m}))$. To check that this is an equilibrium, we are left with (3). On the one hand we have $u(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m)) \leq NW(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m))$ for any $m \in \mathcal{M}$ with

$u(\widehat{q}(m_X^{ESP}), \widehat{q}^I(m_X^{ESP}), m_X^{ESP}, X) = NW(\widehat{q}(m_X^{ESP}), \widehat{q}^I(m_X^{ESP}), m_X^{ESP}, X)$ since $\widehat{q}(m_X^{ESP}) \in [0,1)^K$. On the other hand, we have $NW(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m)) \leq NW(\widehat{q}(m), \widehat{q}^I(m), m_X^{ESP}, X)$ $\leq NW(\widetilde{q}, \widetilde{q}^I, m_X^{ESP}, X)$ for any $m \in \mathcal{M}$. On the whole, we obtain that

$m_X^{ESP} \in \text{Arg}\max_{m \in \mathcal{M}} \; u(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m))$ which thus completes the proof of the existence of an equilibrium where the seller proposes the efficient second-price auction and which implements the first-best.

We next show that any equilibrium that implements the first-best (and thus a fortiori any equilibrium under the PG-refinement) is equivalent to such an equilibrium as derived above.

Consider a given equilibrium $(\widehat{m}, \widehat{q}, \widehat{q}^I, \widehat{\sigma})$ that implements the first-best. We have thus that $u(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m}), \widehat{m}, X; \widehat{\sigma}(\widehat{m})) = NW(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m}), \widehat{m}, X; \widehat{\sigma}(\widehat{m})) = NW(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m}), m_X^{ESP}, X)$

$= \max_{(q,q^I) \in [0,1]^{K+I}} \; NW(q, q^I, m_X^{ESP}, X)$. The last equality implies that $(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m})) \in J(m_X^{ESP}, X)$. From (2), the second equality implies that $W_{N,S}(\widehat{m}, X; \widehat{\sigma}(\widehat{m})) = W_{N,S}(m_X^{ESP}, X)$ for any pair (N, S) that occurs with positive probability, or equivalently that the good is assigned with probability one to the agent with the highest valuation. Besides, any assignment where the good is assigned to the agent with the highest valuation can be implemented with the efficient second-price auction provided that the breaking rule is well-specified (remember that we do not exclude that ties occur with a positive probability since valuation distributions may have some atoms). On the whole we have shown that the equilibrium $(\widehat{m}, \widehat{q}, \widehat{q}^I, \widehat{\sigma})$ is equivalent to an equilibrium where the seller proposes the efficient second-price auction and with the entry rates $(\widehat{q}(\widehat{m}), \widehat{q}^I(\widehat{m})) \in J(m_X^{ESP}, X)$ (an equilibrium which exists thanks to the first part of our proof).

Proof of Lemma 5.1 and Proposition 5.2

In order to show that the (twice continuously differentiable) function $\mu \to NW(\mu, m_X^{ESP}, X)$ is concave on $\mathbb{R}_+^K$, we show that its Hessian matrices, denoted next by $\mathbf{H}_X^\mu$, are semidefinite negative for every $\mu \in \mathbb{R}_+^K$, i.e. that $Y^\top \mathbf{H}_X^\mu Y$ for any vector $Y \in \mathbb{R}^K$ and where $Y^\top$ denotes its transpose (see MasColell et al., 1995, pp. 930-933). Deriving (30), we obtain for any $(k, l) \in \mathcal{K}^2$ that

$$\frac{\partial^2 NW(\mu, m_X^{ESP}, r)}{\partial\mu_k\partial\mu_l} = \sum_{N\in\mathbb{N}^K} P(N|\mu)\cdot\Big[V_{k,[N_{+k}]_{+l}}(m_X^{ESP}) - V_{k,N_{+k}}(m_X^{ESP})\Big]$$
$$= \sum_{N\in\mathbb{N}^K} P(N|\mu)\cdot\Big[\int_X^{\infty}(F^{(1:N_{+l})}(x) - F^{(1:[N_{+k}]_{+l})}(x) + F^{(1:N_{+k})}(x) - F^{(1:N)}(x))dx\Big]$$
$$= -\sum_{N\in\mathbb{N}^K} P(N|\mu)\cdot E_Z\Big[\int_X^{\infty}\prod_{k=1}^{K}[F_k(x|z)]^{n_k}\cdot(1-F_l(x|z))(1-F_k(x|z))dx\Big] \tag{45}$$

Let $Q(x,z) := [(1-F_1(x|z)),\ldots,(1-F_K(x|z))]$. From (45), we have for any $Y\in\mathbb{R}^K$:

$$Y^\top\mathbf{H}_X^\mu Y = -\sum_{N\in\mathbb{N}^K} P(N|\mu)\cdot E_Z\left[\int_X^{\infty}\prod_{k=1}^{K}[F_k(x|z)]^{n_k}\cdot\underbrace{Y^\top\cdot Q(x,z)^\top Q(x,z)\cdot Y}_{=[Q(x,z)Y]^\top\cdot[Q(x,z)Y]\geq 0}dx\right]\leq 0. \tag{46}$$

In other words, (46) says that $\mathbf{H}_X^\mu$ can be viewed as a weighted sum (including integrals) with positive weights of the negative semi-definite matrices $-[Q(x,z)]^\top Q(x,z)$ and is thus also negative semi-definite. This concludes the proof of Lemma 5.1.

In order to show Proposition 5.2, after noting that Proposition 3.2 extends straightforwardly to the Poisson environment, it is sufficient to establish (32). A sufficient condition for this is that $\mu\to NW(\mu, m_X^{ESP}, X)$ is concave on $\mathbb{R}_+^K$ (see MasColell et al., 1995) and we conclude the proof thanks to Lemma 5.1.

Proof of Proposition 6.1

Once the seller is informed about her type, we have to extend our equilibrium concept in Definition 1. Now $\widehat{m}$ should be replaced by a probability distribution over the set of possible mechanisms $\mathcal{M}$ for any possible realization X, denoted by $\widehat{m}(X)$, and (3) should be replaced by

$$\text{Supp}(\widehat{m}(X)) \subseteq \text{Arg}\max_{m\in\mathcal{M}}\ u(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m)) \tag{47}$$

for any possible realization X. Concerning the buyers, they should be equipped with a belief for the types of the seller that announce a given mechanism $m\in\mathcal{M}$. This corresponds equivalently to a distribution, denoted by H_m, over sellers' types. Once a mechanism is proposed by some sellers in equilibrium, then the beliefs should be consistent with the strategy of the seller. For any $m\in\mathcal{M}$ and $k\in\mathcal{K}$, (4) should be replaced by

$$\underset{resp.\ =\binom{0}{1}}{\widehat{q}_k(m)\ \in (0,1)} \Longrightarrow \int u_k(\widehat{q}(m), \widehat{q}^I(m), m; \widehat{\sigma}(m,X))dH_m(X) \underset{resp.\ \binom{\leq}{\geq}}{=} C_k \tag{48}$$

where the way buyers' beliefs matter in their computation of their expected payoff is through the bidding strategy $\widehat{\sigma}(m,X))$ which possibly depends on the realization of the seller's type (since the seller may be allowed to participate in the mechanism). In a mechanism where the seller is inactive at the bidding stage as in standard auction formats (e.g. first- or second-price auctions) without any secret reserve price, then H_m does not play any role in (48). Similarly, for each incumbent $i \in \mathcal{I}$, (5) should be replaced by

$$\widehat{q}_i^I(m) \underset{resp. =\binom{0}{1}}{\in (0,1)} \Longrightarrow \int u_i^I(\widehat{q}(m), \widehat{q}^I(m), m; \widehat{\sigma}(m, X)) dH_m(X) \underset{resp. \binom{\leq}{\geq}}{=} C_i^I. \tag{49}$$

Consider now a given mechanism m such that $\widehat{q}(m) \neq (0, \ldots, 0)$ and that belongs to $\text{Supp}(\widehat{m}(X))$ for at least two realizations x and x' of X. From (48), we obtain the analog of (8), and then the expected objective on the equilibrium path of the seller conditional on having chosen the given mechanism m is smaller than

$\int NW(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m, X)) dH_m(X) \leq \int \left[\max_{\substack{(q,q^I) \in [0,1]^{K+I} \\ m \in \mathcal{M}, \sigma(m) \in \Sigma(m)}} NW(q, q^I, m, X; \sigma(m)) \right] dH_m(X)$ $= \int NW(\widehat{q}(m_X^{ESP}), \widehat{q}^I(m_X^{ESP}), m_X^{ESP}, X) dH_m(X)$ where the last term, which corresponds to the first-best in this environment, corresponds also to the expected objective of the types of the seller choosing m if they were deviating to propose the efficient second-price auction and where the last equality comes from the assumption $(\widehat{q}(m_X^{ESP}), \widehat{q}^I(m_X^{ESP})) \in J(m_X^{ESP}, X)$. If all those sellers were deviating to propose the efficient second-price auction, then the equilibrium conditions (47) impose that they should raise a (weakly) lower objective which implies that the previous inequality hold as an equality. Then we must have $NW(\widehat{q}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m)) = NW(\widehat{q}(m_X^{ESP}), \widehat{q}^I(m_X^{ESP}), m_X^{ESP}, X)$ for $X = x, x'$ with say $x < x'$. This implies further that $(\widehat{q}(m), \widehat{q}^I(m)) \in J(m_x^{ESP}, x) \cap J(m_{x'}^{ESP}, x')$ and thus, from Lemma 3.1, that $(\widehat{q}(m), \widehat{q}^I(m)) \in M(m_x^{ESP}) \cap M(m_{x'}^{ESP})$ and $\widehat{q}(m) \in [0,1)^K$.

Lemma 7.1 *If $x < x'$, then $M(m_x^{ESP}) \cap M(m_{x'}^{ESP}) \cap \{(q, q^I) \in [0,1)^K \times [0,1]^I | q \neq (0, \ldots, 0)\} = \emptyset$.*

Proof Take $(\widetilde{q}, \widetilde{q}^I) \in M(m_x^{ESP}) \cap M(m_{x'}^{ESP}) \cap \{(q, q^I) \in [0,1)^K \times [0,1]^I | q \neq (0, \ldots, 0)\}$ and $k \in \mathcal{K}$ such that $\widetilde{q}_k \neq 0$. From (4), we have then $u_k(\widetilde{q}, \widetilde{q}^I, m_x^{ESP}) = u_k(\widetilde{q}, \widetilde{q}^I, m_{x'}^{ESP}) = C_k$. However, we have $V_{k,N_{+k}}(x) \geq V_{k,N_{+k}}(x')$ for any $N \in \mathbb{N}^K$ with a strict inequality when $N = (0, \ldots, 0)$. Since $\widetilde{q} \in [0,1)^K$, we have $P((0, \ldots, 0)|\widetilde{q}) > 0$, which further implies that $u_k(\widetilde{q}, \widetilde{q}^I, m_x^{ESP}) > u_k(\widetilde{q}, \widetilde{q}^I, m_{x'}^{ESP})$ which raises a contradiction. **Q.E.D.**

We have thus raised a contradiction and there is thus no bunching in equilibrium.

On the equilibrium path, the expected objective of the seller coincides with the expected net welfare which then must coincides with the first-best (otherwise the seller would strictly benefit to deviate and propose the efficient second-price auction). On the whole, for any possible realization of the valuation of the seller, we obtain the equivalence with the efficient second-price auction.

Supplementary Appendix (for online publication)

Generalized second-price auctions

Under exogenous participation, Myerson (1981) shows that the optimal auction can be implemented with a generalized second-price auction where bids are distorted in a very general (nonlinear) way. Similarly, bid distortions play a crucial role in presence of incumbents.

A generalized second-price auction with (general) non-linear distortion (or a bid preference program) is characterized by a reserve price $r \in \mathbb{R}_+$ and a set of right-continuous non-decreasing function, called next bid distortion functions, $A_k : \mathbb{R}_+ \to \mathbb{R}_+$ (for each group $k \in \mathcal{K}$ of potential entrants) and $A_i^I : \mathbb{R}_+ \to \mathbb{R}_+$ (for each incumbent $i \in \mathcal{I}$), which are assumed to be strictly increasing on the set $\{x \in \mathbb{R}_+ | A_k(x) \geq r\}$ for a group k entrant and $\{x \in \mathbb{R}_+ | A_i^I(x) \geq r\}$ for incumbent i. The rules of the generalized second-price auction are as follows:

1. The seller collects all the bids and computes a new (or distorted) bid $A_i^I(b)$ [resp. $A_k(b)$] for each bid b from an incumbent i [resp. from an entrant from group k].

2. The reserve price r is considered next as a bid from the seller.

3. One of the agents (including the seller) with the highest new bid is declared to be the winner and receives the good.[51]

4. Let p denote the maximum of the second highest new bid from participating bidders (if any) and the reserve price r. If the winner is an incumbent i [resp. an entrant from group k], he has to pay $\min\{b \in \mathbb{R}_+ | A_i^I(b) \geq p\}$ [resp. $\min\{b \in \mathbb{R}_+ | A_k(b) \geq p\}$]. The monetary transfer of a buyer who does not receive the good is null.[52]

The price paid by the winner corresponds to the lowest bid he would have to submit in order to be still declared the winner (with some positive probability). Note that the price paid by the winner can never be strictly above his bid. Compared to truthful bidding, bidding below its valuation involves only the loss of some profitable opportunities. Compared to truthful bidding, bidding above its valuation changes the final outcome only in the case where p is above his valuation, i.e. in the events where the final price would have been greater than his valuation. On the whole, we obtain that

Lemma 7.2 *For any generalized second-price auction, truthful bidding is a (weakly) dominant strategy.*

[51]In case of multiple winning bids, we need also a tie-breaking rule to complete the description of a specific auction. Here, any rule would suit, e.g. the one consisting in picking the winner at random.

[52]When there are atoms the tie-breaking rule may matter in terms of the final assignment. Nevertheless, it does not matter in terms of final payoffs in equilibrium since the pricing rule under truthful bidding guarantees that the bidders involved in a tie obtain pay their valuation (this is because $\min\{b \in \mathbb{R}_+ | A_i^I(b) \geq p\} = x$ if $A_i^I(x) = p \geq r$ for any incumbent i while the same holds for entrants).

This further implies that in equilibrium, bidders should bid truthfully in generalized second-price auctions (since we assume that bidders use undominated strategies).

Let $\mathcal{M}_A^{GSP}$ denote the set of generalized second-price auctions with $A_k(b) = b$ for any $k \in \mathcal{K}$. For any $m \in \mathcal{M}_A^{GSP}$, we let $m[r]$ denote the reserve price in the auction and $m[A_i^I]$ the bid distortion of incumbent i, for any $i \in \mathcal{I}$.

Key remark: The virtual pivotal mechanism $m_{\beta,X}^{v\text{-}piv}$ corresponds to the generalized second-price auction in $\mathcal{M}_A^{GSP}$ characterized by $m_{\beta,X}^{v\text{-}piv}[r] = X$ and $m_{\beta,X}^{v\text{-}piv}[A_i^I] = x_i^{virt}(.)$ for any $i \in \mathcal{I}$. We stress that $m_{\beta,X}^{v\text{-}piv}[A_i^I]$ is strictly increasing on the set $\{x \in \mathbb{R}_+ | A_i^I(x) \geq X\}$ thanks to our regularity assumption.

From the perspective of potential entrants, a mechanism $m \in \mathcal{M}_A^{GSP}$ is equivalent to a standard second-price auction with the reserve $m[r]$ and where conditional on z and for any $i \in \mathcal{I}$, the valuation distributions of the incumbents are no longer $F_i^I(.|z)$ but are rather replaced by $F_i^I([m[A_i^I]]^{-1}(.)|z)$ which denotes the distribution of the variable $m[A_i^I](u)$ where the variable u is drawn according to $F_i^I(.|z)$. This results from the fact that their bids are not distorted for $m \in \mathcal{M}_A^{GSP}$. For a given $m \in \mathcal{M}_A^{GSP}$, let $\widetilde{F}_m^{(1:N\cup S)}(x) = E_Z\Big[\prod_{k=1}^K [F_k(x|z)]^{n_k} \cdot \prod_{i\in S} F_i^I([m[A_i^I]]^{-1}(x)|z)\Big]$ denote the CDF of the highest new (or distorted) bid among the bidders under truthful bidding given that the realization of the profile of entrants is N. When m is a (standard) second-price auction, then note we have $\widetilde{F}_m^{(1:N\cup S)} = F^{(1:N\cup S)}$.

With this change of perspective, we obtain on the whole that for any $m \in \mathcal{M}_A^{GSP}$, the expected ex ante utility of an entrant from group k is given by $u_k(q, q^I, m) = \sum_{N\in\mathbb{N}^K} \sum_{S\subseteq\mathcal{I}} P(N|q) \cdot P(S|q^I) \cdot V_{k,N_{+k},S}(m)$ where

$$V_{k,N_{+k},S}(m) = \int_{m[r]}^{\infty} (\widetilde{F}_m^{(1:N\cup S)}(x) - \widetilde{F}_m^{(1:N_{+k}\cup S)}(x))dx. \tag{50}$$

The problem is somehow the same as before from the perspective of entrants, up to the twist that the CDFs of the valuation of the incumbents are now possibly distorted. In particular, (35) still holds which implies that $M(m) \subseteq [0,1)^K \times [0,1]^I$ for any $m \in \mathcal{M}_A^{GSP}$ such that $m[r] = X$. Analogously to the expected welfare function $W_N(m, X; \sigma(m))$, we can define a notion of 'distorted welfare' for any $m \in \mathcal{M}_A^{GSP}$ (which guarantees truthful bidding), denoted by $\widetilde{W}_N(m, X)$, where incumbents' valuations have been substituted by their distorted valuations and the seller's reservation value X by the reserve price $m[r]$. Formally, we define

$$\widetilde{W}_{N,S}(m, X) := \int \max\left\{m[r], \max_{j=1,\dots,|N|}\{x_j^N\}, \max_{i\in S}\{m[A_i^I](x_i^{\mathcal{I}})\}\right\} d[G_{N,S}(y^{N,S})]. \tag{51}$$

We have also $\widetilde{W}_{N,S}(m, X) = m[r] \cdot \widetilde{F}_m^{(1:N\cup S)}(m[r]) + \int_{m[r]}^{\infty} x d[\widetilde{F}_m^{(1:N\cup S)}(x)]$.

The fundamental property of the pivotal mechanism (11) translates now into

$$\widetilde{W}_{N_{+k},S}(m, X) - \widetilde{W}_{N,S}(m, X) = V_{k,N_{+k},S}(m) \text{ for any } k \in \mathcal{K} \text{ and } m \in \mathcal{M}_A^{GSP} \tag{52}$$

In words, entrants obtain the incremental surplus they generate where the surplus is defined according to the distorted valuations.

For any $m \in \mathcal{M}_A^{GSP}$, we let $\widetilde{NW}(q, q^I, m, X) := \sum_{N \in \mathbb{N}^K} \sum_{S \subseteq \mathcal{I}} P(N|q) \cdot P(S|q^I) \cdot \widetilde{W}_{N,S}(m, X) - \sum_{k=1}^K q_k \mathcal{N}_k \cdot C_k - \sum_{i=1}^I \beta_i^I \cdot q_i^I \cdot C_i^I$ denote the total expected (ex ante) distorted welfare.

Comment: For the virtual pivotal mechanism, namely when $m = m_{\beta,X}^{v\text{-}piv}$, then the terms $\widetilde{W}_{N,S}(m, X)$, $\widetilde{NW}(q, q^I, m, X)$ are equal to $W_{N,S}^{virt}(m, X)$, $NW^{virt}(q, q^I, m, X)$.

As in (40), we get for each $k \in \mathcal{K}$ that

$$\frac{\partial \widetilde{NW}(q, q^I, m, X)}{\partial q_k} = \mathcal{N}_k \cdot \left(\sum_{N \in \mathbb{N}^K} \sum_{S \subseteq \mathcal{I}} P_k(N|q) \cdot P(S|q^I) \cdot [\widetilde{W}_{N_{+k},S}(m, X) - \widetilde{W}_{N,S}(m, X)] - C_k \right). \quad (53)$$

Combined with (52), we get then that

$$\frac{\partial \widetilde{NW}(q, q^I, m, X)}{\partial q_k} = \mathcal{N}_k \cdot \Big[u_k(q, q^I, m, X) - C_k \Big]. \quad (54)$$

Concerning incumbents, we have for any $i \in S$

$$V_{i,N,S}^I(m) = E_Z \Big[\int_{m[r]}^{\infty} \int_0^u ([m[A_i^I]]^{-1}(u) - [m[A_i^I]]^{-1}(\max\{y, m[r]\})) \cdot d[\widetilde{F}_m^{(1:N \cup S_{-i})}(x|z)] \widetilde{f}_i^I(u|z) du \Big]. \quad (55)$$

If the function $x - m[A_i^I](x)$ is decreasing, then we obtain that

$$\begin{aligned} V_{i,N,S}^I(m) &\leq E_Z \Big[\int_{m[r]}^{\infty} \int_0^u (u - \max\{y, m[r]\}) \cdot d[\widetilde{F}_m^{(1:N \cup S_{-i})}(x|z)] \widetilde{f}_i^I(u|z) du \Big] \\ &= \int_{m[r]}^{\infty} (\widetilde{F}_m^{(1:N \cup S_{-i})}(x) - \widetilde{F}_m^{(1:N \cup S)}(x)) dx \\ &= \widetilde{W}_{N,S}(m, X) - \widetilde{W}_{N,S_{-i}}(m, X). \end{aligned} \quad (56)$$

Thanks to the regularity assumption, the function $x - m[A_i^I](x)$ is decreasing for any $i \in \mathcal{I}$ in the virtual pivotal mechanism and we get thus (26).

As in (42), we get for each $i \in \mathcal{I}$ that

$$\frac{\partial \widetilde{NW}(q, q^I, m, X)}{\partial q_i^I} = \sum_{N \in \mathbb{N}^K} \sum_{\substack{S \subseteq \mathcal{I} \\ i \in S}} P(N|q) \cdot \prod_{i' \in S_{-i}} q_{i'}^I \cdot \prod_{i' \in \mathcal{I} \setminus S} (1 - q_{i'}^I) \cdot \Big[\widetilde{W}_{N,S}(m, X) - \widetilde{W}_{N,S_{-i}}(m, X) \Big] - C_i^I. \quad (57)$$

Combined with (26), we get then for the virtual pivotal mechanism that

$$\frac{\partial \widetilde{NW}(q, q^I, m^{v\text{-}piv}, X)}{\partial q_i^I} \geq u_i^I(q, q^I, m^{v\text{-}piv}, X) - C_i^I. \quad (58)$$

The Poisson model: Adapting in a straightforward way the notation we introduced in the binomial model when there are incumbents, we have in the Poisson model with incumbents that $\mu \to \widetilde{NW}(\mu, q^I, m, X)$ is concave on $\mathbb{R}_+^K$ for any $m \in \mathcal{M}_A^{GSP}$ with $m[r] = X$ and so in particular for either the virtual pivotal mechanism or the efficient second-price auction. This holds because:

$$\frac{\partial \widetilde{NW}(\mu, q^I, m, X)}{\partial \mu_k} = \sum_{N\in\mathbb{N}^K} \sum_{S\subseteq\mathcal{I}} P(N|\mu)\cdot P(N|q^I)\cdot V_{k,N_{+k},S}(m) - C_k \tag{59}$$

which implies then from (50) that

$$\frac{\partial^2 \widetilde{NW}(\mu, q^I, m, X)}{\partial \mu_k \partial \mu_l} = -\sum_{N\in\mathbb{N}^K} \sum_{S\subseteq\mathcal{I}} P(N|\mu)\cdot P(S|q^I) E_Z\left[\int_X^\infty \prod_{k=1}^K [F_k(x|z)]^{n_k}\cdot \prod_{i\in S} F_i^I([m[A_i^I]]^{-1}(x)|z)\cdot(1-F_l(x|z))(1-F_k(x|z))dx\right]. \tag{60}$$

Proof of Proposition 4.1

The proof is almost the same as the one of Proposition 3.2. The difference is that due to the rents of the incumbents, a calculation à la Myerson (1981) comes on the top of it such that we have to deal with the virtual net total welfare instead of the net total welfare.

From a classic calculation using the Envelope Theorem (see Myerson 1981)[53] and given that from (14) the distribution of $y_{-i}^{N,S}$ conditional on x_i^S coincides with the unconditional distribution $G_{-i,N,S}(y_{-i}^{N,S})$, we have that for any equilibrium

$$\frac{dV_i^I(x, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m))}{dx} = \sum_{N\in\mathbb{N}^K} \sum_{S\subseteq\mathcal{I}_{-i}} P(N|\widehat{q}(m))\cdot P_i(S|\widehat{q}^I(m))\cdot E_{y_{-i}^{N,S}|x_i^S=x}\left[Q_{i,N,S}^I(y^{N,S};\widehat{\sigma}(m))\right] \tag{61}$$

and then by integration

$$\begin{aligned} &V_i^I(x, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m)) = V_i^I(\underline{x}_i, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m)) \\ &+\textstyle\sum_{N\in\mathbb{N}^K}\sum_{S\subseteq\mathcal{I}_{-i}} P(N|\widehat{q}(m))\cdot P_i(S|\widehat{q}^I(m))\cdot \int_{\underline{x}_i}^x \int Q_{i,N,S}^I(y^{N,S};\widehat{\sigma}(m))d[G_{-i,N,S}(y_{-i}^{N,S})]dx_i^S \end{aligned} \tag{62}$$

for any $x \geq \underline{x}_i$ and any $m \in \mathcal{M}$ and then by integration over x and with an integration per parts

$$\begin{aligned} &u_i^I(\widehat{q}(m), \widehat{q}^I(m), m; \widehat{\sigma}(m)) = V_i^I(\underline{x}_i, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m)) \\ &+\textstyle\sum_{N\in\mathbb{N}^K}\sum_{S\subseteq\mathcal{I}_{-i}} P(N|\widehat{q}(m))\cdot P_i(S|\widehat{q}^I(m))\cdot \int \frac{1-F_i^I(x_i^{\mathcal{I}})}{f_i^I(x_i^{\mathcal{I}})} Q_{i,N,S}^I(y^{N,S};\widehat{\sigma}(m))d[G_{N,S}(y^{N,S})]. \end{aligned} \tag{63}$$

Summing those rents, we get the expression (15) for the rents of the seller's objective.

For each $i \in \mathcal{I}$, note that the participation constraints at the auction stage reduce to

$$V_i^I(\underline{x}_i, m; \widehat{q}(m), \widehat{q}^I(m), \widehat{\sigma}(m)) \geq 0, \tag{64}$$

while the incentive compatibility constraints require that the function

[53] We stress that the fact that the incumbents may receive additional information other than just their private valuation, e.g. about the variable z, does not change the argument.

$$x \to \sum_{N\in\mathbb{N}^K}\sum_{S\subseteq\mathcal{I}_{-i}} P(N|\widehat{q}(m))\cdot P_i(S|\widehat{q}^I(m))\cdot\int Q^I_{i,N,S}((x,y^{N,S}_{-i});\widehat{\sigma}(m))d[G_{-i,N,S}(y^{N,S}_{-i})] \quad \text{is non-decreasing on} \quad [\underline{x}_i,\overline{x}_i], \tag{65}$$

a constraint that will not be binding next thanks to the 'regularity' assumption which guarantees that the virtual pivotal mechanism belongs to $\mathcal{M}^{GSP}_A$ as shown previously.

For a given $(\widetilde{q},\widetilde{q}^I)\in J^{virt}(m^{v\text{-}piv}_{\beta,X},X)\cap\mathcal{Q}^{ex}$ satisfying A5, we let $\widehat{m}=m^{v\text{-}piv}_{\beta,X}$, $(\widehat{q}(m),\widehat{q}^I(m))=(\widetilde{q},\widetilde{q}^I)$ and $\widehat{\sigma}(m)$ be the truthful strategy if $m=m^{v\text{-}piv}_{\beta,X}$ and we pick any undominated strategy $\sigma(m)\in\Sigma(m)$ and any participation rates $(\widehat{q}(m),\widehat{q}^I(m))\in M(m;\sigma(m))$ for $m\in\mathcal{M}\setminus\{m^{v\text{-}piv}_{\beta,X}\}$ (which is possible since $M(m;\sigma(m))\neq\emptyset$). From (28), we obtain that $(\widehat{q}(\widehat{m}),\widehat{q}^I(\widehat{m}))\in M(\widehat{m};\widehat{\sigma}(\widehat{m}))$. To check that this is an equilibrium, we are left with (3). On the one hand we have $u(\widehat{q}(m),\widehat{q}^I(m),m,X;\widehat{\sigma}(m))\leq NW^{virt}(\widehat{q}(m),\widehat{q}^I(m),m,X;\widehat{\sigma}(m))$ for any $m\in\mathcal{M}$ with $u(\widehat{q}(m^{v\text{-}piv}_{\beta,X}),\widehat{q}^I(m^{v\text{-}piv}_{\beta,X}),m^{v\text{-}piv}_{\beta,X},X)$
$=NW^{virt}(\widehat{q}(m^{ESP}_X),\widehat{q}^I(m^{ESP}_X),m^{ESP}_X,X)$ since $\widehat{q}(m^{ESP}_X)\in[0,1)^K$ (see eq. (19) and the sequel). On the other hand, we have $NW^{virt}(\widehat{q}(m),\widehat{q}^I(m),m,X;\widehat{\sigma}(m))\leq NW^{virt}(\widehat{q}(m),\widehat{q}^I(m),m^{v\text{-}piv}_{\beta,X},X)$
$\leq NW^{virt}(\widetilde{q},\widetilde{q}^I,m^{v\text{-}piv}_{\beta,X},X)$ for any $m\in\mathcal{M}$ (see eq. (25) and the definition of $J^{virt}(m^{v\text{-}piv}_{\beta,X},X)$). On the whole, we obtain that $m^{v\text{-}piv}_{\beta,X}\in\operatorname{Arg\,max}_{m\in\mathcal{M}}\ u(\widehat{q}(m),\widehat{q}^I(m),m,X;\widehat{\sigma}(m))$ which guarantees that this is an equilibrium and thus completes the proof of the existence of an equilibrium where the seller proposes the virtual pivotal mechanism and which implements the virtual first-best.

We next show that any equilibrium that implements the virtual first-best (and thus a fortiori any equilibrium with $\widehat{q}^I(m^{v\text{-}piv}_{\beta,X})=\widetilde{q}^I$ under the PG-refinement) is equivalent to such an equilibrium as derived above.

Consider a given equilibrium $(\widehat{m},\widehat{q},\widehat{q}^I,\widehat{\sigma})$ that implements the virtual first best. We have thus that $u(\widehat{q}(\widehat{m}),\widehat{q}^I(\widehat{m}),\widehat{m},X;\widehat{\sigma}(\widehat{m}))=NW^{virt}(\widehat{q}(\widehat{m}),\widehat{q}^I(\widehat{m}),\widehat{m},X;\widehat{\sigma}(\widehat{m}))=NW^{virt}(\widehat{q}(\widehat{m}),\widehat{q}^I(\widehat{m}),m^{v\text{-}piv}_{\beta,X},X)$
$=\max_{(q,q^I)\in[0,1]^{K+I}}\ NW^{virt}(q,q^I,m^{v\text{-}piv}_{\beta,X},X)$. The last equality implies that $(\widehat{q}(\widehat{m}),\widehat{q}^I(\widehat{m}))\in J^{virt}(m^{v\text{-}piv}_{\beta,X},X)$. From (2), the second equality implies that $W^{virt}_{N,S}(\widehat{m},X;\widehat{\sigma}(\widehat{m}))=W^{virt}_{N,S}(m^{v\text{-}piv}_{\beta,X},X)$ for any pair (N,S) that occurs with positive probability, or equivalently that the good is assigned with probability one to the agent with the highest virtual valuation. Besides, any assignment where the good is assigned to the agent with the highest valuation can be implemented with the virtual pivotal mechanism provided that the breaking rule is well-specified (remember that we do not exclude that ties occur with a positive probability). On the whole we have shown that the equilibrium $(\widehat{m},\widehat{q},\widehat{q}^I,\widehat{\sigma})$ is equivalent to an equilibrium where the seller proposes the virtual pivotal mechanism and with the entry rates $(\widehat{q}(\widehat{m}),\widehat{q}^I(\widehat{m}))\in J^{virt}(m^{v\text{-}piv}_{\beta,X},X)$ (an equilibrium which exists thanks to the first part of our proof).

Extension of Proposition 5.2 to multi-object auctions

The analysis without incumbents in Section 3 and 5 extends in a straightforward way with multiple heterogenous objects when buyers' valuations and the seller's reservation values are both additive across objects, a model where the "pivotal" mechanism corresponds to using the efficient second-price auction for each objects. The key element in the argument still consists in showing that the net total welfare function is concave as a function of the vector of entry in the Poisson model. The net first part of the total welfare (i.e. the one which comes only from the assignment of the goods) can be viewed as a sum of the expressions where each object would have been treated separately. In particular, the only thing that matters are the marginal distributions for each object. As a sum of concave functions, this first term remains concave with multiple objects. The second part of the total welfare (i.e. the one capturing the entry costs) is linear with respect to μ which concludes the argument for concavity.

Remark: At first glance, such a setup seems to resume to a sum of various setups with a single good for sale. This is not completely the case since we allow implicitly some economies of scale through the entry costs. However those terms in the net total welfare function are linear in the vector of entry which does not alter the concavity property.

The analysis extends also to multi-unit auctions when buyers have unit-demand and the seller has flat reservation values. The key element in the argument still consists in showing that the net total welfare function is concave as a function of the vector of entry as it is established below. Formally, consider that the seller has L identical units of a good and assume that her reservation value for each unit equals X. The generalization of the standard second-price auction with the reserve price r is the $L+1^{th}$-price auction with the reserve price r. When $r = X$, this corresponds precisely to the "pivotal mechanism", i.e. the mechanism that match bidders' rents with their contribution to the welfare. To alleviate the notation, we show this point without incumbents and when entrants' valuations are drawn independently. But the result generalizes in a straightforward way to a setup with incumbents and with conditionally independent valuations.

A buyer with valuation $u \geq r$ who participates in the $L+1^{th}$-price auction with the reserve price r against the profile N when the reserve price is r will receive the expected payoff of $\int_r^u F^{(L:N)}(x)dx$. Sticking to our previous notation, the expected payoff of a group k buyer from entering such an auction (i.e. before knowing the realization of his valuation) is thus given by

$$V_{k,N_{+k}}(X) = \int_X^{\infty} F^{(L:N)}(x)(1 - F_k(x))dx \tag{66}$$

which is a generalization of (34). We have then

$$\begin{aligned} V_{k,[N_{+k}]_{+l}}(X) - V_{k,N_{+k}}(X) &= \int_X^{\infty} (F^{(L:N_{+l})}(x) - F^{(L:N)}(x))(1 - F_k(x))dx \\ &= \int_X^{\infty} (F_l(x) \cdot F^{(L:N)}(x) + (1 - F_l(x)) \cdot F^{(L-1:N)}(x) - F^{(L:N)}(x))(1 - F_k(x))dx \\ &= -\int_X^{\infty} \underbrace{(F^{(L:N)}(x) - F^{(L-1:N)}(x))}_{\geq 0} \cdot (1 - F_l(x))(1 - F_k(x))dx \end{aligned} \tag{67}$$

We obtain then a kind of generalized version of (46)

$$Y^\top \mathbf{H}_X^\mu Y = -\sum_{N\in\mathbb{N}^K} P(N|\mu)\cdot\left[\int_X^\infty \left[F^{(L:N)}(x) - F^{(L-1:N)}(x)\right]\cdot Y^\top\cdot(Q(x)^\top Q(x))\cdot Y dx\right] \leq 0, \tag{68}$$

where $\mathbf{H}_X^\mu$ is the Hessian matrix of the net total welfare function at μ in the pivotal mechanism and where $Q(x) := [(1-F_1(x)),\dots,(1-F_K(x))]$.

With respect to this setup, the allocation problem in the sponsored search auction setup (that has been presented in Section 5) can be decomposed as assigning first L homogenous units of size s_L where each bidder can receive at most one unit, second $L-1$ homogenous units of size $s_{L-1}-s_L$, and so on the last stage being a single unit of size s_1-s_2. Sticking to our previous notation, we let $V_{k,N_{+k}}(X)$ denote the expected payoff of a group k buyer from entering the pivotal mechanism associated to the reservation value X per unit of good for the seller and facing the profile N of entrants. The generalization of (34) is now

$$V_{k,N_{+k}}(X) = \sum_{l=1}^{L} s_l\cdot\int_X^\infty F^{(l:N)}(x)(1-F_k(x))dx. \tag{69}$$

We see thus that the problem shrinks to a linear combination of the previous one. The generalized version of (68) is then

$$Y^\top \mathbf{H}_X^\mu Y = -\sum_{N\in\mathbb{N}^K} P(N|\mu)\big[\int_X^\infty \sum_{l=1}^{L} s_l\cdot\left[F^{(l:N)}(x) - F^{(l-1:N)}(x)\right]\cdot Y^\top\cdot(Q(x)^\top Q(x))\cdot Y dx\big] \leq 0, \tag{70}$$

with $Q(x) := [(1-F_1(x)),\dots,(1-F_K(x))]$ and with the convention $F^{(0:N)}(x) = 0$.

The Poisson model and its foundation

To define the equilibrium formally in the Poisson model, we have to slightly adapt our notation. We let

- $\mu = (\mu_1,\dots,\mu_K) \in [0,\infty)^K$ denote the profile of entry rates of potential entrants, namely when the Poisson distribution of group k buyer has mean μ_k for any $k\in\mathcal{K}$.
- $P(N|\mu) = P_k(N|\mu) = e^{-\sum_{k'=1}^K \mu_{k'}}\cdot\prod_{k'=1}^K \frac{[\mu_{k'}]^{n_{k'}}}{n_{k'}!}$ denote the probability of both the realization $N\in\mathbb{N}^K$ for the set of entrants and the realization $N\in\prod_{k=1}^K[0,\mathcal{N}_k]$ for the set of opponents of a given entrant from group k, when the profile of entry probabilities for potential entrants is μ.

Then all the expressions of the expected ex ante utilities of the various agents extend by replacing q and $P(N|q)$ with μ and $P(N|\mu)$ respectively. We have e.g. that $u(\mu,q^I,m,X;\sigma(m)) = \sum_{N\in\mathbb{N}^K}\sum_{S\subseteq\mathcal{I}} P(N|\mu)\cdot P(S|q^I)\cdot\Lambda_{N,S}(m,X;\sigma(m))$ denote the expected (ex ante) utility of the seller with valuation X in the mechanism m when the profile of entry rates is μ for potential entrants and q_I for incumbents and when buyers follow the bidding profile $\sigma(m)$.

For technical reasons, we add an additional constraint for the set of possible mechanisms $\mathcal{M}^*$ in the Poisson model: we assume that the monetary transfers of all agents (both the buyers and the seller) are bounded by some amount $\overline{T} > 0$. This restriction is a purely technical trick to define equilibria properly, in particular to avoid problems that could arise in unbounded mechanisms.

To define the equilibrium formally, for each $k \in \mathcal{K}$ we introduce as a counterpart to the binomial parameter functions $\widehat{q}_k : \mathcal{M} \to [0,1]$ a Poisson parameter function $\widehat{\mu}_k : \mathcal{M} \to R_+$, where $\widehat{\mu}_k(m)$ characterizes the distribution of participation of buyers of type k in the mechanism m. An equilibrium in the Poisson model is then defined as:

Definition 4 *For a given set of possible mechanisms $\mathcal{M} \subseteq \mathcal{M}^*$, an equilibrium in the Poisson model of entry is defined as a strategy profile $(\widehat{m}, (\widehat{\mu}_k)_{k\in\mathcal{K}}, (\widehat{q}_i^I)_{i\in\mathcal{I}}, \widehat{\sigma})$, where $\widehat{m} \in \mathcal{M}$ stands for the seller's chosen mechanism, $\widehat{\mu}_k : \mathcal{M} \to R_+$ [resp. $\widehat{q}_i^I : \mathcal{M} \to [0,1]$] describes the Poisson entry rates of group k buyers [resp. the entry probability of incumbent i] in the various possible mechanisms $m \in \mathcal{M}$, and $\widehat{\sigma}(m) \in \Sigma(m)$ describes the bidding profile of the bidders in $m \in \mathcal{M}$ such that*

1. *(Utility maximization for the seller)*

$$\widehat{m} \in Arg \max_{m\in\mathcal{M}} \; u(\widehat{\mu}(m), \widehat{q}^I(m), m, X; \widehat{\sigma}(m)). \tag{71}$$

2. *(Utility maximization for group k buyers at the entry stage, for any $k \in \mathcal{K}$) for any $m \in \mathcal{M}$,*

$$\widehat{\mu}_k(m) \underset{resp.\ =0}{>0} \Longrightarrow u_k(\widehat{\mu}(m), \widehat{q}^I(m), m; \widehat{\sigma}(m)) \underset{resp.\ \leq}{=} C_k. \tag{72}$$

3. *(Utility maximization for incumbent i at the entry stage, for any $i \in \mathcal{I}$) for any $m \in \mathcal{M}$,*

$$\widehat{q}_i^I(m) \underset{resp.\ =\binom{0}{1}}{\in (0,1)} \Longrightarrow u_i^I(\widehat{\mu}(m), \widehat{q}^I(m), m; \widehat{\sigma}(m)) \underset{resp.\ \binom{\leq}{\geq}}{=} C_i^I. \tag{73}$$

4. *(Equilibrium conditions at the bidding stage) in any mechanism $m \in \mathcal{M}$, bidders are using undominated strategies. Furthermore, when the seller chooses the mechanism $\widehat{m}$, the bidding profile $\widehat{\sigma}(\widehat{m})$ forms a Bayes-Nash equilibrium given the entry profile $(\widehat{\mu}(\widehat{m}), \widehat{q}^I(\widehat{m}))$.*

The notion of equivalence between two strategy profiles become:

Definition 5 *In the Poisson model, we say that two strategy profiles $(m, \{\mu_k\}_{k\in\mathcal{K}}, \{q_i^I\}_{i\in\mathcal{I}}, \{\sigma(m)\}_{m\in\mathcal{M}})$ and*
$(\widetilde{m}, \{\widetilde{q}_k\}_{k\in\mathcal{K}}, \{\widetilde{q}_i^I\}_{i\in\mathcal{I}}, \{\widetilde{\sigma}(m)\}_{m\in\mathcal{M}})$ are equivalent if the profile of entry rates/probabilities at the mechanism proposed by the seller are the same, namely $\mu(m) = \widetilde{\mu}(\widetilde{m})$ and $q^I(m) = \widetilde{q}^I(\widetilde{m})$, and

if for any profile of bidders (N, S) that occurs with positive probability,[54] *then the good is assigned in the same way with probability one (which implies in particular that $W_{N,S}(m, X; \sigma(m)) = W_{N,S}(\widetilde{m}, X; \widetilde{\sigma}(m))$).*

From the same arguments as in Proposition 3.3, we have:

Lemma 7.3 *Assume that $m_X^{ESP} \in \mathcal{M}$ and A2. Any equilibrium in the Poisson model that implements the first-best is equivalent to an equilibrium where the seller proposes the efficient second-price auction.*

Comment: Under A3 and A4 and if $m_{\beta,X}^{v-piv} \in \mathcal{M}$, then we have the result analogous to Proposition 4.1: Any equilibrium in the Poisson model that implements the virtual first-best is equivalent to an equilibrium where the seller proposes the virtual pivotal mechanism (and with the same entry profile for the incumbents).

The analysis with a finite set of potential entrants extends to the Poisson model: as detailed in Subsection 5.1, the revenue of the seller is equal to the first-best when the seller proposes the efficient second-price auction and for any entry profile $(\widehat{\mu}, \widehat{q}^I) \in J(m_X^{ESP}, X)$ which are equilibrium profiles since we still have $J(m_X^{ESP}, X) \subseteq M(m_X^{ESP})$. What remains to be shown in order to guarantee that it is an equilibrium is that $M(m; \widehat{\sigma}(m)) \neq \emptyset$ for any $m \in \mathcal{M}^*$. Furthermore, we will delineate the status of the equilibria in the Poisson model by showing that any limit (in a sense that will be made precise in the sequel) of a sequence of equilibria in the binomial model should implement the first-best and thus be equivalent to an equilibrium in which the seller proposes the efficient second-price auction.

Comment: All our arguments for environments without incumbents (where the efficient second-price auction implements the first-best) extend straightforwardly to environments with incumbents as in Section 4 and where the virtual pivotal mechanism implements the virtual first-best. In particular, we would obtain that any limit of a sequence of equilibria implements the virtual first-best.

A preliminary mathematical lemma Consider two sequences $(q_n)_{n\in\mathbb{N}} \in [0, 1]^{\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}} \in [-K, K]^{\mathbb{N}}$ with $K \in R_+$. For any $n \in \mathbb{N}$, we define

$$S[n] := \sum_{i=0}^{n} \binom{n}{i} [q_n]^i [1 - q_n]^{n-i} \cdot A_i. \tag{74}$$

Lemma 7.4 *If $n \cdot q_n$ converges to $\mu < \infty$ when n goes to infinity, then*

$$\lim_{n\to\infty} S[n] = \sum_{i=0}^{\infty} e^{-\mu} \frac{\mu^i}{i!} \cdot A_i \in [-K, K]. \tag{75}$$

[54]Formally, $P(N|\mu(m)) \cdot P(S|q^I(m)) > 0$.

Proof Let $\overline{\mu} = \max_{n\in\mathbb{N}} n \cdot q_n < \infty$ which is well defined since $n \cdot q_n$ converges to $\mu < \infty$. For any $z \in \mathbb{N}$ with $z \leq n$, we define $S_1[n,z] := \sum_{i=0}^{z-1} \binom{n}{i}[q_n]^i[1-q_n]^{n-i} \cdot A_i$ and $S_2[n,z] := \sum_{i=z}^{n} \binom{n}{i}[q_n]^i[1-q_n]^{n-i} \cdot A_i$. We have thus $S[n] = S_1[n,z] + S_2[n,z]$.

For any pair $i, z \in \mathbb{N}$, we have $(i+z)! \geq i!z!$, which further implies that $\binom{n}{i+z} \leq \frac{n^z}{z!}\binom{n-z}{i}$ for any $n \geq i+z$. After some calculation we have then for any pair $n, z \in \mathbb{N}$ with $z \leq n$, $|S_2[n,z]| = |\sum_{i=0}^{n-z} \binom{n}{i+z} \cdot [q_n]^{i+z}[1-q_n]^{n-i-z} \cdot A_i| \leq \sum_{i=0}^{n-z} \frac{[n \cdot q_n]^z}{z!}\binom{n-z}{i} \cdot [q_n]^i[1-q_n]^{n-i-z} \cdot |A_i| \leq \frac{\overline{\mu}^z}{z!} \cdot K \cdot \sum_{i=0}^{n-z} \binom{n-z}{i} \cdot [q_n]^i[1-q_n]^{n-i-z} = \frac{\overline{\mu}^z}{z!} \cdot K$. It is well-known from the properties of the factorial that $\lim_{n\to\infty} \frac{\overline{\mu}^z}{z!} = 0$. We have also $|\sum_{i=z}^{\infty} e^{-\mu}\frac{\mu^i}{i!} \cdot A_i| \leq \frac{\mu^z}{z!} \cdot |\sum_{i=0}^{\infty} e^{-\mu}\frac{\mu^i}{i!} \cdot A_{i+z}| \leq \frac{\mu^z}{z!} \cdot K$. For any $\epsilon > 0$, we can thus pick z large enough (say $z = z^*$) such that $\max_{n\in\mathbb{N}:n\geq z^*} |S_2[n,z^*]| \leq \frac{\epsilon}{3}$ and $|\sum_{i=z^*}^{\infty} e^{-\mu}\frac{\mu^i}{i!} \cdot A_i| \leq \frac{\epsilon}{3}$.

For any $i < z^*$, we have $\lim_{n\to\infty} [1-q_n]^{n-i} = e^{-\mu}$ and then we can easily check that $\lim_{n\to\infty} \binom{n}{i}[q_n]^i[1-q_n]^{n-i} = e^{-\mu}\frac{\mu^i}{i!}$.

Consequently, if n is large enough, $|S_1[n,z^*] - \sum_{i=0}^{z^*-1} e^{-\mu}\frac{\mu^i}{i!} \cdot A_i|$ can be bounded by $\frac{\epsilon}{3}$ and then finally $|S[n] - \sum_{i=0}^{\infty} e^{-\mu}\frac{\mu^i}{i!} \cdot A_i| \leq |S_1[n,z^*] - \sum_{i=0}^{z^*-1} e^{-\mu}\frac{\mu^i}{i!} \cdot A_i| + |S_2[n,z^*]| + |\sum_{i=z^*}^{\infty} e^{-\mu}\frac{\mu^i}{i!} \cdot A_i|$ is bounded by ϵ. Since this is true for any $\epsilon > 0$, we have established (75). **Q.E.D.**

For a given vector of potential entrants $\mathcal{N}$, let $\widehat{m}[\mathcal{N}]$, $\widehat{q}[\mathcal{N}]$ and $\widehat{\sigma}[\mathcal{N}]$ denote the equilibrium played. We let also $\overline{NW}[\mathcal{N}]$ denote the corresponding equilibrium net total welfare, namely $\overline{NW}[\mathcal{N}] := \sum_{N\in\mathbb{N}^K} P(N|\widehat{\mu}[\mathcal{N}]) \cdot W_N(\widehat{m}[\mathcal{N}], X; \widehat{\sigma}(m)[\mathcal{N}]) - \sum_{k=1}^{K} \widehat{q}_k[\mathcal{N}] \cdot \mathcal{N}_k \cdot C_k$.

To develop "limit" results, we consider sequences of the form $(\mathcal{N}[l])_{l\in\mathbb{N}}$ where $\mathcal{N}[l] = (\mathcal{N}_1[l], \ldots, \mathcal{N}_K[l]) \in \mathbb{N}^K$ such that for any $k \in \mathcal{K}$, $\mathcal{N}_k[l]$ goes to infinity when l goes to infinity. When l goes to infinity, it is our way to formalize that the number of potential entrants goes large in each group.

The following proposition gives the status of considering the Poisson model: any limit of equilibria in the binomial model (under the conditions of Proposition 3.2) implements the first-best in the Poisson model.

Proposition 7.5 *Assume that $m_X^{ESP} \in \mathcal{M}$ and A2. When l goes to infinity, $\overline{NW}[\mathcal{N}[l]]$ goes to $\max_{\mu\in R_+^K} NW(\mu, m_X^{ESP}, X)$, namely the equilibrium net total welfare goes to the first-best total welfare in the Poisson model when the number of potential participants goes to infinity in the binomial model.*

Proof As a preliminary, we show the following lemma for any given $m \in \mathcal{M}^*$:

Lemma 7.6 *Consider that a given strategy σ^* is played in equilibrium in the mechanism m for any $\mathcal{N}$.*[55] *If for any $k \in \mathcal{K}$, $\mathcal{N}_k[l] \cdot \widehat{q}_k(m)[\mathcal{N}[l]]$ goes to $\mu_k(m) \in R_+ \cup \{\infty\}$ when l goes to infinity, then $\mu_k(m) < \infty$ for each k and $\mu(m) \in M(m; \sigma^*)$.*

In words, any limit of equilibrium entry probabilities in the binomial model is an equilibrium profile in the Poisson model.

Proof of Lemma 7.6 Below, we use the following notation for each $k \in \mathcal{K}$: $\mathcal{N}_j^k = \mathcal{N}_k$ for $j \in \mathcal{K} \setminus \{j\}$ and $\mathcal{N}_k^k = \mathcal{N}_k - 1$. Consider a given k. We first establish that $\mu_k(m) < \infty$. Suppose by

[55] This is a very strong assumption. However, in generalized second-price auctions, we have in mind that σ^* corresponds to truthful bidding in the sequel.

contradiction that $\mu_k(m) = \infty$. Then there exists l^* such that $l \geq l^*$ implies that $\widehat{q}_k(m)[\mathcal{N}[l]] > 0$ and then from (4), we obtain

$$\sum_{n_1=0}^{\mathcal{N}_1^k[l]} \cdots \sum_{n_K=0}^{\mathcal{N}_K^k[l]} \prod_{j=1}^{K} \binom{\mathcal{N}_j^k[l]}{n_j} [\widehat{q}_j(m)[\mathcal{N}[l]]]^{n_j} [1-\widehat{q}_j(m)[\mathcal{N}[l]]]^{\mathcal{N}_j^k[l]-n_j} \cdot V_{k,N_{+k}}(m;\sigma^*) \underset{(resp.\ \geq)}{=} C_k \text{ if } \widehat{q}_k(m)[\mathcal{N}[l]] \underset{(resp.\ =)}{<} 1,$$

for any $l \geq l^*$. This is also equivalent to

$$\sum_{n_1=0}^{\mathcal{N}_1^k[l]} \cdots \sum_{n_K=0}^{\mathcal{N}_K^k[l]} \prod_{j=1}^{K} \binom{\mathcal{N}_j^k[l]}{n_j} [\widehat{q}_j(m)[\mathcal{N}[l]]]^{n_j} [1-\widehat{q}_j(m)[\mathcal{N}[l]]]^{\mathcal{N}_j^k[l]-n_j} \cdot n_k \cdot V_{k,N_{+k}}(m;\sigma^*) \underset{(resp.\ \geq)}{=} \mathcal{N}_k[l] \cdot \widehat{q}_k(m)[\mathcal{N}[l]] \cdot C_k \quad (76)$$

if $\widehat{q}_k(m)[\mathcal{N}[l]] < 1$ [resp. $\widehat{q}_k(m)[\mathcal{N}[l]] = 1$], for any $l \geq l^*$. Since transfers and valuations are bounded we have then that the sum of the payoffs of any subset of agents is bounded by $\overline{x} + \overline{T}$ for any realization of the set of entrants. We have thus in particular that $n_k \cdot V_{k,N_{+k}}(m) \leq \overline{x} + \overline{T}$ for any N. The left-hand term in (76) is the expectation over N of an expression that is uniformly bounded by $\overline{x} + \overline{T}$ is also uniformly bounded by $\overline{x} + \overline{T}$ while the right-hand term goes to infinity, which leads to a contradiction. We have thus $\mu_k(m) < \infty$.

By repeated use of Lemma 7.4 for each of the K sums, we obtain then for each $k \in \mathcal{K}$ that the fact that the inequality

$$\sum_{n_1=0}^{\mathcal{N}_1^k[l]} \cdots \sum_{n_K=0}^{\mathcal{N}_K^k[l]} \prod_{j=1}^{K} \binom{\mathcal{N}_j^k[l]}{n_j} [\widehat{q}(m)[\mathcal{N}[l]]]^{n_j} [1-\widehat{q}(m)[\mathcal{N}[l]]]^{\mathcal{N}_j^k[l]-n_j} \cdot V_{k,N_{+k}}(m;\sigma^*) \leq C_k$$

holds for any $l \in \mathbb{N}$ implies that
$\sum_{N \in \mathbb{N}^K} e^{-\sum_{j=1}^K \mu_j(m)} \prod_{j=1}^K \frac{[\mu_j(m)]^{n_j}}{n_j!} V_{k,N_{+k}}(m;\sigma^*) \leq C_k$. Suppose now that $\mu_k(m) > 0$. Then there exists l^* such that $l \geq l^*$ implies that $\widehat{q}_k(m)[\mathcal{N}[l]] > 0$ and then from (4), we obtain then that the equality
$\sum_{n_1=0}^{\mathcal{N}_1^k[l]} \cdots \sum_{n_K=0}^{\mathcal{N}_K^k[l]} \prod_{j=1}^K \binom{\mathcal{N}_j^k[l]}{n_j} [\widehat{q}_j(m)[\mathcal{N}[l]]]^{n_j} [1-\widehat{q}_j(m)[\mathcal{N}[l]]]^{\mathcal{N}_j^k[l]-n_j} \cdot V_{k,N_{+k}}(m;\sigma^*) = C_k$ holds for any $l \geq l^*$ which further implies that
$\sum_{N \in \mathbb{N}^K} e^{-\sum_{j=1}^K \mu_j(m)} \prod_{j=1}^K \frac{[\mu_j(m)]^{n_j}}{n_j!} V_{k,N_{+k}}(m;\sigma^*) = C_k$. We have then established that $\mu_k(m)$ satisfies the equilibrium equation (72) for any k. On the whole we have $\mu(m) \in M(m;\sigma^*)$. **End of the proof of Lemma 7.6**

The rest of the proof goes as follows. In the limit Poisson model, we know all equilibria implement the first-best. Thus, we cannot have accumulation points of the sequence of welfare for the finite economy that are away from the first-best, as otherwise it would imply that in the Poisson model, some net welfare other than the first-best could be achieved. Formally, consider the following sequence of entry rates: in equilibrium in the binomial model with $\mathcal{N}$, when the seller proposes the efficient second-price auction, the entry probabilities are given by $\widehat{q}(m_X^{ESP})[\mathcal{N}]$ and let $NW^*[\mathcal{N}]$ be the corresponding total welfare in the binomial model (which may be strictly lower than the first-best solution given by (10). Next the first-best welfare is denoted by $NW^{opt}[\mathcal{N}]$). Note that we necessarily have $\widehat{q}_k(m_X^{ESP})[\mathcal{N}] \cdot \mathcal{N}_k \leq \frac{\overline{x}}{C_k}$ because the expected revenue of the seller should be larger than X (and so must be the net total welfare in equilibrium). We show below that the sequence $NW^*[\mathcal{N}[l]]$ converges to $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$. To establish this, we show that every subsequence has a subsequence that converges to $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$.

Every subsequence has a subsequence such that $\widehat{q}_k(m_X^{ESP})[\mathcal{N}[l]] \cdot \mathcal{N}_k[l]$ converges to some μ_k^* for each k (because $\widehat{q}_k(m_X^{ESP})[\mathcal{N}[l]] \cdot \mathcal{N}_k[l]$ stays in the compact set $[0, \frac{\overline{x}}{C_k}]$). Then we can apply Lemma 7.6 (where the strategy σ^* in Lemma 7.6 corresponds to truthful bidding here) and we obtain that $\mu^* \in M(m_X^{ESP})$ and then $\mu^* \in J(m_X^{ESP}, X)$ (because $M(m_X^{ESP}) = J(m_X^{ESP}, X)$ in the Poisson model). Then we can apply Lemma 7.4 iteratively for each k to the corresponding subsequence of the net welfare $(NW^*[\mathcal{N}[l']])$,[56]

and we obtain that there exist a subsequence whose total welfare has the limit $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$, namely the first-best in the Poisson model. With the same argument, we have also that the sequence $(NW^{opt}[\mathcal{N}[l]])_{l \in \mathbb{N}}$ converges to $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$ when l goes to infinity.

Let us come back to the core of the proof of Proposition 7.5. Since the expected net total welfare is bounded in equilibrium ($\overline{NW}[\mathcal{N}] \in [X, \overline{x}]$), it is sufficient to show that the sequence $(\overline{NW}[\mathcal{N}[l]])_{l \in \mathbb{N}}$ cannot have another accumulation point.

Suppose that an accumulation point of the sequence $\overline{NW}[\mathcal{N}[l]]$ lies strictly below

$\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$, then it would raise a contradiction since it would imply that we can pick a large enough l such that the revenue of the seller is strictly below the one it would have raised with the efficient second-price auction (because we have shown above that the net total welfare, or equivalently the revenue of the seller in equilibrium when the number of potential in each is large enough (such that A1 is satisfied), converges to $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$) when l goes to infinity.

We have that $\overline{NW}[\mathcal{N}] \leq \overline{NW}^{opt}[\mathcal{N}]$ for any $\mathcal{N}$. Since we have shown above that the sequence $(NW^{opt}[\mathcal{N}[l]])_{l \in \mathbb{N}}$ converges to $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$) when l goes to infinity, then we obtain as corollary that any accumulation point of the sequence $(\overline{NW}[\mathcal{N}[l]])_{l \in \mathbb{N}}$ is below $\max_{\mu \in R_+^K} NW(\mu, m_X^{ESP}, X)$. On the whole, the only possible accumulation point of the sequence $(\overline{NW}[\mathcal{N}[l]])_{l \in \mathbb{N}}$ is the first-best welfare in the Poisson model. **Q.E.D.**

To conclude this section, let us show that there exist an equilibrium within the Poisson model.

For a given mechanism $m \neq m_X^{ESP}$, fix a given undominated strategy $\widehat{\sigma}(m)$. Then for any $\mathcal{N}$, we can define corresponding equilibrium entry profiles in the binomial model, say $\widehat{q}_k(m; \widehat{\sigma}(m))[\mathcal{N}]$. Note that we have shown that $\mathcal{N}_k \cdot \widehat{q}_k(m; \widehat{\sigma}(m))[\mathcal{N}] \in [0, \frac{\overline{X+T}}{C_k}]$ for any binomial model $\mathcal{N}$. For

[56] Note that we have:

$$NW^*[\mathcal{N}[l]] = \sum_{n_1=0}^{\mathcal{N}_1[l]} \cdots \sum_{n_K=0}^{\mathcal{N}_K[l]} \prod_{k=1}^{K} \binom{\mathcal{N}_k[l]}{n_k} [\widehat{q}_k(m_X^{ESP})[\mathcal{N}[l]]]^{n_k} [1 - \widehat{q}_k(m_X^{ESP})[\mathcal{N}[l]]]^{\mathcal{N}_k[l] - n_k} \cdot W_N(m_X^{ESP}, X) - \sum_{k=1}^{K} \widehat{q}_k(m_X^{ESP})[\mathcal{N}[l]] \cdot \mathcal{N}_k[l] \cdot C_k.$$

any sequence $(\mathcal{N}[i])_{i\in\mathbb{N}}$, there is a subsequence $(\mathcal{N}[i])_{\sigma(i)\in\mathbb{N}}$ ($\sigma:\mathbb{N}\to\mathbb{N}$ is an increasing function) such that $\mathcal{N}_k[\sigma(i)]\cdot\widehat{q}_k(m;\widehat{\sigma}(m))[\mathcal{N}[\sigma(i)]]$ goes to $\mu(m)\in[0,\frac{\overline{X}+T}{C_k}]^K$. From Lemma 7.6, we obtain that $\mu(m)\in M(m;\widehat{\sigma}(m))$ and thus that $M(m;\widehat{\sigma}(m))\neq\emptyset$. As a corollary, the construction we did to build an equilibrium that implements the firsts-best in the binomial model carries over for the Poisson model. In particular, the set of equilibria according to Definition 4 is not empty.

Discrimination with linear distortions

Let $\mathcal{M}_X^{linear}\subseteq\mathcal{M}_A^{GSP}$ denote the set of generalized second-price auctions with $r=X$, $A_k(b)=\alpha\cdot b$ if $b\geq r$ and $A_k(b)=0$ otherwise for any $k\in\mathcal{K}$, and $A_i^I(b)=\alpha^I\cdot b$ if $b\geq r$ and $A_i^I(b)=0$ otherwise for any $i\in\mathcal{I}$, with both $\alpha\geq 1$ and $\alpha^I\geq 1$. It is straightforward to check that both the allocation rule and the payment rule depend solely on the ratio $r^\alpha=\frac{\alpha}{\alpha^I}\in(0,\infty)$ and so that any generalized second-price auction in $\mathcal{M}_X^{linear}$ is characterized solely by this ratio. We use next the notation $r_\alpha\in(0,\infty)$ to denote a generic mechanism in $\mathcal{M}_X^{linear}$. The mechanism $r_\alpha=1$ corresponds to the efficient second-price auction.

We consider the Poisson model throughout this Section. We will also consider that the entry costs of the incumbents are null so that they enter with probability one for any $m\in\mathcal{M}_X^{linear}$ and that their rents are not fully internalized by the seller, i.e. A3. We also assume homogenous entrants ($K=1$). Last we assume that if the efficient second-price auction is posted, then some potential entrants will enter with positive probability. To sum up,

Assumption A 7 *$K=1$, $C_i^I=0$ for each $i\in\mathcal{I}$, and $u(0,(1,\cdots,1),m_X^{ESP})>C$.*

To alleviate notation, for any $r_\alpha\in\mathcal{M}_X^{linear}$, we let then $V_{1,N,\mathcal{I}}(r_\alpha;\sigma(r_\alpha))\equiv V_n(r_\alpha)$, $V_{i,N,\mathcal{I}}^I(r_\alpha;\sigma(r_\alpha))\equiv V_{i,n}^I(r_\alpha)$, $u_k(\mu,(1,\cdots,1),r_\alpha;\sigma(r_\alpha))\equiv u_k(\mu,r_\alpha)$ and $u_i^I(\mu,(1,\cdots,1),r_\alpha;\sigma(r_\alpha))\equiv u_i^I(\mu,r_\alpha)$.

To alleviate the proof, we also add this technical assumption:

Assumption A 8 *The CDFs $F(.|z)$ and $F_i^I(.|z)$ do not depend on z and are continuously differentiable on their (common) support $[\underline{x},\overline{x}]$ with $X\in(\underline{x},\overline{x})$.*

In the auction $r_\alpha\in\mathcal{M}_X^{linear}$, the probability of an entrant [resp. the incumbent $i\in\mathcal{I}$] with valuation $x\geq X$ to win the good when he faces $n-1$ competing entrants [resp. n entrants] and when the set of incumbents that participate is $\mathcal{I}$ is equal to $[F(x)]^{n-1}\cdot\prod_{i=1}^I F_i^I(\max\{r_\alpha\cdot x,X\})\equiv F_{r_\alpha}^{1:(n-1)\cup\mathcal{I}}(x)$ [resp. $[F(\max\{\frac{x}{r_\alpha},X\})]^n\cdot\prod_{\substack{i'=1\\ i'\neq i}}^I F_{i'}^I(x)\equiv F_{r_\alpha}^{1:n\cup\mathcal{I}_{-i}}(x)$]. Using standard results from auction theory, we obtain that the (interim) payoff of an entrant [resp. the incumbent i] from participating in the auction r_α when facing $n-1$ competing entrants [resp. n entrants] and when the set of incumbents that participate is $\mathcal{I}$ is given by

$$V_n(r_\alpha)=\int_X^{\overline{x}}F_{r_\alpha}^{1:(n-1)\cup\mathcal{I}}(x)\cdot(1-F(x))dx\ \ \text{[resp. } V_{i,n}^I(r_\alpha)=\int_X^{\overline{x}}F_{r_\alpha}^{1:n\cup\mathcal{I}_{-i}}(x)\cdot(1-F_i^I(x))dx\ \text{]}.\quad(77)$$

The ex ante expected payoff of an entrant [resp. the incumbent i] from participating in the auction r_α with the participation rate μ is then given by

$$u(\mu, r_\alpha) = \int_X^{\overline{x}} e^{-\mu(1-F(x))} \prod_{i=1}^{I} F_i^I(\max\{r_\alpha \cdot x, X\})(1-F(x))dx \tag{78}$$

[resp. $u_i^I(\mu, r_\alpha) = \int_X^{\overline{x}} e^{-\mu(1-F(\max\{\frac{x}{r_\alpha}, X\}))} \prod_{i'=1, i'\neq i}^{I} F_{i'}^I(x)(1-F_i^I(x))dx$]. Note that $u_i^I(\mu, r_\alpha) > 0$ for any μ and r_α which justifies that we assumed above that incumbents participate to the auction with probability 1 (more generally any bidders with a valuation strictly above X has always a strictly positive expected profit because he may face no competitors with a valuation above r). We have then

$$\frac{\partial u(\mu, r_\alpha)}{\partial r_\alpha} = r_\alpha \cdot \sum_{i=1}^{I} \int_{\max\{\frac{X}{r_\alpha}, X\}}^{\overline{x}} e^{-\mu(1-F(x))} \prod_{\substack{i'=1 \\ i'\neq i}}^{I} F_i^I(r_\alpha \cdot x)(1-F(x))f_i^I(r_\alpha \cdot x)dx \geq 0 \tag{79}$$

where the inequality is strict if $r_\alpha \in (\frac{X}{\overline{x}}, \frac{\overline{x}}{X})$, and

$$\frac{\partial u_i^I(\mu, r_\alpha)}{\partial r_\alpha} = -\frac{\mu}{[r_\alpha]^2} \int_{\max\{r_\alpha \cdot X, X\}}^{\overline{x}} e^{-\mu(1-F(\frac{x}{r_\alpha}))} \prod_{\substack{i'=1 \\ i'\neq i}}^{I} F_{i'}^I(x)(1-F_i^I(x))f(\frac{x}{r_\alpha})dx \leq 0 \tag{80}$$

where the inequality is strict if $r_\alpha \in (\frac{X}{\overline{x}}, \frac{\overline{x}}{X})$ and $\mu > 0$. We have similarly:

$$\frac{\partial u(\mu, r_\alpha)}{\partial \mu} = -\int_X^{\overline{x}} e^{-\mu(1-F(x))} \prod_{i=1}^{I} F_i^I(\max\{r_\alpha \cdot x, X\})(1-F(x))^2 dx < 0 \tag{81}$$

and

$$\frac{\partial u_i^I(\mu, r_\alpha)}{\partial \mu} = -\int_X^{\overline{x}} e^{-\mu(1-F(\max\{\frac{x}{r_\alpha}, X\}))} \prod_{\substack{i'=1 \\ i'\neq i}}^{I} F_{i'}^I(x)(1-F_i^I(x))(1-F(\max\{\frac{x}{r_\alpha}, X\}))dx \leq 0 \tag{82}$$

where the last inequality is strict if $r_\alpha > \frac{X}{\overline{x}}$. From (81), the equilibrium condition (72) has a unique solution when the posted mechanism is r_α: the solution corresponds thus to $\widehat{\mu}(r_\alpha)$ the equilibrium entry rate for potential entrants when the posted mechanism is r_α.

Lemma 7.7 *We have* $\frac{d\widehat{\mu}(r_\alpha)}{dr_\alpha} \geq 0$ *for any point* $r_\alpha \in (0, \infty)$ *such that* $\widehat{\mu}(r_\alpha) > 0$. *As a corollary, the function* $r_\alpha \to \widehat{\mu}(r_\alpha)$ *is nondecreasing.*

Proof Deriving the equilibrium condition (72) at r_α such that $\widehat{\mu}(r_\alpha) > 0$, we have $\frac{d\widehat{\mu}(r_\alpha)}{dr_\alpha} = -\frac{\frac{\partial u(\widehat{\mu}(r_\alpha), r_\alpha)}{\partial r_\alpha}}{\frac{\partial u(\widehat{\mu}(r_\alpha), r_\alpha)}{\partial \mu}} \geq 0$. We conclude with the inequalities (79) and (81). **Q.E.D.**

Favoring entrants with respect to incumbents has two impacts on the informational rents of the incumbents: on the one hand, raising the ratio r_α reduces their informational rents ceteris paribus (eq. (80)); on the other hand, raising r_α increases the incentives of the potential entrants to enter the auction which is detrimental indirectly to the incumbents because they face more competition from new entrants (eq. (82)). On the whole, it is thus not ambiguous that increasing r_α is detrimental to the incumbents.

Lemma 7.8 *For each incumbent $i \in \mathcal{I}$, we have $\frac{du_i^I(\widehat{\mu}(r_\alpha), r_\alpha)}{dr_\alpha} \leq 0$ and the inequality is strict if $r_\alpha = 1$.*

Proof We note first that $\frac{du_i^I(\widehat{\mu}(r_\alpha), r_\alpha)}{dr_\alpha} = \underbrace{\frac{\partial u_i^I(\widehat{\mu}(r_\alpha), r_\alpha)}{\partial r_\alpha}}_{\substack{\leq 0 \text{ with} \\ <0 \text{ if } r_\alpha \in (\frac{X}{\overline{x}}, \frac{\overline{x}}{X})}} + \underbrace{\frac{d\widehat{\mu}(r_\alpha)}{dr_\alpha}}_{\geq 0} \cdot \underbrace{\frac{\partial u_i^I(\widehat{\mu}(r_\alpha), r_\alpha)}{\partial \mu}}_{\leq 0} \leq 0.$

We conclude by noting that $1 \in (\frac{X}{\overline{x}}, \frac{\overline{x}}{X})$. **Q.E.D.**

Next proposition formalizes that incumbents should be discriminated against entrants.

Proposition 7.9 *Assume A3, A7, A8, and $\mathcal{M} = \mathcal{M}_X^{linear}$. In equilibrium, the chosen mechanism r_α satisfies $r_\alpha > 1$.*

Proof of Proposition 7.9 In equilibrium, the revenue of the seller is given by

$$u(\widehat{\mu}(r_\alpha), r_\alpha) = \left[\sum_{n=0}^{\infty} e^{-\widehat{\mu}(r_\alpha)} \frac{[\widehat{\mu}(r_\alpha)]^n}{n!} \cdot W_{N,\mathcal{I}}(r_\alpha, X) - \widehat{\mu}(r_\alpha) \cdot C\right] - \sum_{i=1}^{I} (1 - \beta_i^I) \cdot u_i^I(\widehat{\mu}(r_\alpha), r_\alpha). \quad (83)$$

Furthermore, the term in the bracket corresponds to the total net welfare which is maximized at $r_\alpha = 1$. Combined with Lemma 7.8, we conclude that $\operatorname{Arg\,max}_{r_\alpha \in (0,\infty)} u(\widehat{\mu}(r_\alpha), r_\alpha) \subseteq (1, \infty)$. **Q.E.D.**