

On the benefits of set-asides

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Abstract

Set-asides programs consist in forbidding access to specific participants, and they are commonly used in procurement auctions. We show that when the set of potential participants is composed of an incumbent (who bids for sure if allowed to) and of entrants who show up endogenously (in such a way that their expected rents are fixed by outside options), then it is always beneficial for revenues to exclude the incumbent in the second-price auction. This exclusion principle is generalized to auction formats that favor the incumbent in the sense that he would always gets the good when he values it most. By contrast, set-asides need not be desirable if the incumbent's payoff is included into the seller's objective or in environments with multiple incumbents. Various applications are discussed.

Keywords: set-asides, entry restrictions, auctions with endogenous entry, entry deterrence, asymmetric buyers, incumbents, government procurement, procurement competition policy.

JEL classification: D44, H57, L10, L40.

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1 Introduction

Set-asides are used extensively in government procurements and resource sales like spectrum auctions. They consist either in excluding from the auction some specific bidders (e.g. large firms or previous auction winners)^{1,2} or equivalently in limiting the access to some well-chosen bidders (e.g. domestic firms, small firms or new entrants). Specifically, set-asides are routinely used to favor small businesses in countries like the US, Canada or Japan. In Japan, for example, approximately two thirds of civil engineering contracts are subject to set-asides (Nakabayashi, 2013). In the US, those federal procurement contracts billing between \$3,000 and \$150,000 are automatically reserved to small and medium enterprises (SMEs) (Kang and Miller, 2016) while for larger contracts, the Federal Acquisition Regulations (FAR) leave some discretion to contracting officers to exclude one or more sources, in particular under the motives that it would “*increase or maintain competition and likely result in reduced overall costs for the acquisition, or for any anticipated acquisition*”.³ Promoting small businesses is also an important goal in the EU, but the European economic law is attached to principles of transparency, non-discrimination and equal treatment that prohibit explicit discriminatory practices.⁴ Even when explicit discrimination is not possible, implicit set-asides can still be at work for example through the requirement of technological constraints that would be known to forbid the access of some potential participants or by excluding a bidder on the ground that he previously caused some disappointment in the past (see Saussier and Tirole (2015) who discuss some new European directives in this vein).

An important practical question is whether, when and to what extent set-asides policies reduce the costs in public procurements.⁵ The research question addressed in this paper is whether excluding some bidders could reduce the procurement cost, or equivalently boost the seller’s revenue in auction contexts. When addressing this question, we have in mind a positive perspective by which we mean that we take the auction format as given and we investigate the effect of set-asides. This is in contrast with the normative perspective seeking for the optimal auction format

¹When contracts are renewed periodically, we could consider whether it could be beneficial to exclude the incumbent. E.g. in the Veolia Transport and Transdev case, the French antitrust authorities impose as a remedy that Transdev Group, the merged entity, commits not to bid in a bunch of cities in the south-east of France for which it was the incumbent (see paragraphs 446-450 in “*Décision n°10-DCC-198 du 30 décembre 2010*” and also paragraph 43 in “*Décision n°13-DCC-137 du 1^{er} octobre 2013*” that confirms that Transdev Group has met his engagements). We thank Thibaud Vergé for bringing this antitrust case to our attention.

²In the auctions that determine the subsidy for the production of renewable energy in Portugal, a winning bidder is automatically excluded from participating in the subsequent auctions (del Rio (2016)).

³See FAR 6.2 (Full and Open Competition After Exclusion of Sources): https://www.acquisition.gov/far/html/Subpart%206_2.html. Kang and Miller (2016) report that among the contracts that are auctioned, one third is subject to exclusion.

⁴The winner should be the most advantageous tender while exclusion grounds are very limited (DIRECTIVE 2014/24/EU, Article 57).

⁵There is a limited empirical literature on pro-small business set-asides (Denes (1997), Nakabayashi (2013) and Athey, Coey and Levin (2013)) which finds evidence that the induced increasing participation from small businesses more than compensates the loss from those who have been set aside. On the other hand, this literature found mixed evidence concerning the procurement costs/auction revenues. Beyond the competition motive, Kang and Miller (2016) provide evidence that set-asides can save on the administrative costs attached to the processing of bids. Coviello et al. (2016) argue that the possibility to exclude bidders with bad reputation allows to reduce moral hazard issues. They find evidence that set-asides can increase the probability that the same firm wins repeatedly the contract as well as enhance a variety of ex post performance measure of quality (such as reductions in delays).

as formalized by Myerson (1981) in contexts with exogenous participation or more recently by Jehiel and Lamy (2015) in contexts with endogenous entry. The main objective of this paper is to contribute to the policy debate on set-asides, assuming other discriminatory tools are not available.

In the context of single-good second-price auctions with private values, when the set of participants is exogenous, excluding some bidders reduces competition, and thus set-asides can only be detrimental to the seller (excluding some bidders can only reduce the second-highest valuation among the bidders and thus the final price).⁶ By contrast, when participation is endogenous, it would seem set-asides may possibly boost the participation of potential entrants in such a way that it is beneficial to the seller. In this spirit, Cramton (2013) argues that set-asides were at the core of the success of Canada's spectrum auctions for Advanced Wireless Services (AWS) because they encouraged participation from deep-pocketed new entrants, thereby resulting in a push of the prices both for the blocks with set-asides and the blocks without. Relatedly, in the UK 2000's spectrum auctions, the fact that one licence was reserved to a new entrant is considered to be an important source of its success.⁷ The logic behind those cases is that set-asides can be pro-competitive insofar as not allowing some bidder(s) to participate can boost the participation of other kinds of bidders and possibly be overall beneficial.

We start our analysis with the second-price auction in which the reserve price is set at the seller's valuation - a format that we refer to as the Vickrey auction. We develop an endogenous entry model in which bidders are either incumbents who participate for sure in the auction or potential entrants whose participation rate is endogenously determined to ensure that the expected payoff entrants derive from participating matches their outside options. In some parts, we solve for the optimal set-aside policy determining whom from the incumbents or which groups of entrants should be banned. In other parts, we restrict the question to studying the benefit of excluding a specific incumbent or a specific group of potential entrants.

Our first general insight is that irrespective of the shape of the distributions of valuations

⁶When bidders' valuations are i.i.d., Bulow and Klemperer (1996) provide a bound on how excluding one bidder is detrimental to the seller in an efficient auction: They show that switching to the optimal mechanism (namely the second-price auction with optimally set reserve price) would not compensate the seller from losing a single bidder. For the first-price auction with independently distributed valuations, Arozamena and Cantillon (2004) show that when the valuation distribution of a bidder is upgraded then his competitors bid collectively more aggressively in equilibrium. As a corollary, it implies that the seller's revenue decreases when some bidders are excluded. Under affiliated private values, Pinkse and Tan (2005) show on the contrary that a bidder may not bid more aggressively when the number of its competitors increases. However, they do not address the incidence of increased competition on the seller's revenue. Menicucci (2009) builds an example with symmetric bidders where the seller's revenue is the largest when competition is limited to two bidders.

⁷In addition to the four incumbents, eleven new entrants participated in the UK auction, which strongly contrasts with the failures in terms of both entry and revenue in the auctions for the 3G Telecom licences organized by some other European countries (see Binmore and Klemperer (2002) and Jehiel and Moldovanu (2003)), such as the Netherlands where only one entrant showed up resulting in a revenue per capita that was four times smaller than in UK. Maybe the most drastic example of a harmful impact of an incumbent on competition is when a single bidder deters all potential entrants from participating as illustrated in the 1994 US spectrum auctions where a license covering all of southern California was offered and where it was publicly known that Pacific Bell, the incumbent company, had a higher valuations than its rivals and thus should win for sure in an ascending auction (see Milgrom (2004) for details and Klemperer (2002) for other entry deterrence examples, in particular with first price auctions).

and the magnitude of entry costs, when there is only one incumbent, it is always beneficial for revenues to exclude the incumbent in the Vickrey auction.⁸ Excluding the incumbent is clearly seen to be desirable when the incumbent is so strong that his presence would discourage any other participation, thereby resulting in a rather poor revenue. But, our insight turns out to hold no matter how strong the incumbent is. The logic for this can be understood as follows. From Jehiel and Lamy (2015), we know that when the incumbent is out, the seller's revenue corresponds (in expectation) to the welfare net of the entry costs of the entrants -referred to as the total welfare- and that the best possible revenue is the one obtained through the Vickrey auction. When the incumbent is in, revenues are reduced for the following reason. Making use of a fundamental property of the Vickrey auction, the incumbent gets a rent equal to his marginal contribution to the welfare. Hence, by a simple accounting argument, the seller's revenue corresponds (in expectation) to the total welfare as if the incumbent were absent. Because participation rates are optimally determined when the incumbent is out (as shown in Jehiel and Lamy, 2015), but typically not so when the incumbent participates, we conclude that excluding the incumbent is always good for revenues. Thus, when there is only one incumbent, the indirect benefit of excluding the incumbent obtained through a boost of entrants' participation always dominates the direct cost of not having the incumbent for a fixed set of participants.

Interestingly, the insight about the desirability of excluding the incumbent carries over to other auction formats, as long as the incumbent gets a payoff no smaller than his marginal contribution to the welfare. Using mechanism design techniques (that allow us to relate bidders' expected payoffs to the allocation rule), we show that this is so when the good is over-assigned to the incumbent in the sense that he always gets the good when he values it most. Such an observation allows us to extend our exclusion principle beyond the Vickrey auction and to cover practically important applications. For example, in the context of procurement auctions where the incumbent is better at renegotiating the contract than entrants (see Bajari, Houghton and Tadelis (2014) for a discussion of how important renegotiations are in procurement auctions) then we show that excluding the incumbent enhances the revenues.

When the seller's objective is the revenues augmented by the incumbent's payoff, we show that the seller never finds it profitable to exclude the incumbent in auctions that do not assign him the good when he does not value it most. Thus, when the incumbent is more reliable than the entrants (maybe due to asymmetric risks of breakdown), we obtain as a corollary that no-exclusion is optimal for the seller in the Vickrey auction if she internalizes the incumbent's payoff. In a similar vein, we show in environments without incumbents but two groups of entrants differing among other things in their reliability that the seller's revenue decreases if the more reliable group is set aside and if the auction format does not discriminate along that dimension. This result is a corollary of a more general insight for environments with two groups of potential entrants saying roughly that if one group is disadvantaged in the auction while the other is advantaged,

⁸We note that such a finding is of practical importance to the extent that in a dynamic perspective the incumbent can often be thought of as the current contractor and there is typically one of them.

then excluding the disadvantaged group decreases the seller's revenue.

Finally, we briefly discuss the desirability of exclusion in the context of first-price auctions and how our general results could be applied. In environments with a single incumbent and when the latter has a right of first refusal allowing him to match the bids of his competitors, we show that excluding the incumbent increases the seller's revenue. In standard first-price auctions (i.e. without right-of-first-refusal), we observe that excluding the incumbent is good when the incumbent and entrants' valuations are drawn independently from the same distribution (and a fortiori so when the incumbent's valuation is drawn from a weaker distribution). By contrast, if the incumbent's valuation is drawn from a stronger distribution and if the seller's internalizes the incumbent's payoff, then no-exclusion is optimal for the seller.

When moving to multiple incumbents, the determination of the optimal set-asides policy is more complex, even in the context of the Vickrey auction. Indeed, when there are multiple incumbents, excluding an incumbent may have an extra detrimental effect not present in the one incumbent case on the rent left to the other incumbents. When the incumbents' valuation distributions can be interpreted as arising from collusive rings of entrants of random sizes, we show that the exclusion of any incumbent has no effect on the rent left to other incumbents (due to the adjustment of entrants' participation decisions), and thus it is optimal to exclude all incumbents in this case. For some range of alternative distributional assumptions, we show that excluding an incumbent increases the rents left to the other incumbents, and we observe then that excluding a sufficiently weak incumbent would be detrimental to revenues.

The rest of the paper is organized as follows. Section 2 presents our general model with endogenous entry and illustrates some of our main results through simple examples. Section 3, after introducing some formal notation, establishes that in the Vickrey auction, no-exclusion is always optimal from a welfare perspective. Section 4 establishes our main result when there is a single incumbent and the auction format is the Vickrey auction. Section 5 extends the insight obtained with one incumbent to other (possibly ex post inefficient) formats. Section 6 moves to environments with multiple incumbents. Section 7 briefly considers the case of interdependent values and whether split awards and entry fees may be preferable to set-asides. Section 8 concludes. Technical proofs are gathered in the Appendix.

2 An auction setup with set-asides

2.1 The model

A risk neutral seller S is selling a good through a given auction procedure. Her reservation value is denoted by $X_S \geq 0$. When the good is auctioned off through a second-price auction with a reserve price set at X_S , then the auction format is referred to as the Vickrey auction. We assume that there are two classes of buyers: The incumbents who participate for sure in the auction (alternatively, we can think of their participation costs as being null) and the potential entrants

who can submit a bid only if they have incurred an entry cost. The decisions to participate are made simultaneously by all potential entrants. We allow entrants to come from various groups, and different groups can be characterized both by different entry costs and different distributions of valuations. The number of potential entrants is assumed to be large in each group, which allows us to simplify the analysis (as explained below).

Remark. It should be mentioned that our model is framed as an auction setup in which buyers are bidding in order to acquire an object. It could obviously be phrased as a procurement in which the designer seeks to obtain a service from various potential providers. The procurement interpretation fits better some of our motivations and policy implications developed in Introduction. It can be observed that for contracts of services that are auctioned off periodically, a bidder we refer to as an incumbent may be thought of as being the previous holder of the contract, which will motivate our study of the single incumbent scenario.⁹

Formally, the various possible buyers and their distributions of valuations and entry costs are described as follows. There is a finite set of incumbents \mathcal{I} . Each incumbent $i \in \mathcal{I}$ is characterized by a cumulative distribution $F_i^I(\cdot|z)$, from which his valuation is drawn conditional on the realization z of some underlying variable Z .¹⁰ There are K groups of potential entrants $\mathcal{E} = \{1, \dots, K\}$. Each group is composed of infinitely many potential buyers, which will justify our modelling of entry as following Poisson distributions (see below). A buyer from group $k \in \mathcal{E}$ has an entry cost $C_k > 0$ and his valuation is drawn from the cumulative distribution $F_k(\cdot|z)$ conditional on the realization z of the underlying variable Z .¹¹ Conditionally on z , the valuations of the various buyers are assumed to be drawn independently.¹² To simplify, we also assume that the supports of the distributions $F_i^I(\cdot|z)$ and $F_k(\cdot|z)$ are uniformly bounded from above by $\bar{x} > X_S$, but we make no other assumption on these distributions nor on the distribution of z . We will describe what bidders observe and at which stage later on.

While our general formulation puts no restriction on the number of groups,¹³ some of our results require further restrictions. We say that potential entrants are symmetric if $K = 1$ (and then we drop the index k from our notation). We say that all buyers (entrants and incumbents alike) are symmetric if $F_i^I(\cdot|z) = F_k(\cdot|z) = F(\cdot|z)$ for each $i \in \mathcal{I}$ and $k \in \mathcal{E}$.¹⁴ For some results concerning auctions in which there is no weakly dominant strategy, we impose the additional

⁹In an auction perspective, scarce resources like spectrum may also be auctioned periodically in a way that makes the incumbency status relevant.

¹⁰We assume implicitly that the variable Z belongs to a measurable space and is distributed according to some probability measure, so as to ensure that the integrals and expectations to be introduced next are well-defined.

¹¹The entry cost C_k can be interpreted equivalently as the expected utility of a group k buyer if he chooses an outside option (which may consist e.g. in participating in another procurement).

¹²Given that we impose no specific structure on the variable Z , conditional independence is a general way to introduce some correlation between buyers' valuations.

¹³This contrasts with the structural empirical literature that has typically considered two-group cases as e.g. in Athey, Levin and Seira (2011) and Athey, Coey and Levin (2013). The group structure can thus capture the idea of pre-entry signals about valuations as in Roberts and Sweeting (2012) or Gentry and Li (2014). The special case in which $F_k(x|z) = \mathbf{1}[x > x_k]$ for any $k \in \mathcal{E}$ corresponds to the case in which potential entrants from group k know their valuation x_k before entry as in McAfee (1993).

¹⁴Levin and Smith (1994) consider a model with symmetric potential entrants and no incumbents.

restriction that valuations are drawn independently with a common support $[\underline{x}, \bar{x}]$ according to continuously differentiable distributions (over their common support). We refer to such environments as *Myersonian environments*.¹⁵

In this paper, we adopt the view that the only instrument of the seller is the *set-asides policy*. Formally,

Definition 1 *A set-asides policy is a pair (I, E) which designates the set of incumbents $I \subseteq \mathcal{I}$ and the groups of entrants $E \subseteq \mathcal{E}$ who are allowed to participate. (I, E) is announced before the buyers decide whether or not to participate.*

Special cases of set-asides policies include: 1) $(I, E) = (\mathcal{I}, \mathcal{E})$, which corresponds to no-exclusion, and 2) $(I, E) = (\emptyset, \mathcal{E})$, which corresponds to excluding all the incumbents. For some results, we derive the optimal set-asides policy as if the seller were free to choose any (I, E) . For other results, we consider instead whether excluding some specific bidders can be profitable for the seller (and to motivate the latter, we have implicitly in mind that the seller would not be allowed to exclude some types of buyers). It should be noted that our main results do not rely on the ability to exclude entrants.

Our main interest in the rest of the paper will be to understand the effect of (I, E) on the revenue generated by the seller.

The timing of the game is as follows. The auction format is exogenously given. The seller announces first her set-asides policy (I, E) . Second, potential entrants decide simultaneously whether or not to participate. At that time, the only information potential entrants have is the group they come from. Third, participants learn their private valuations so that we are in a private value environment (see Section 7.1 for an extension to an interdependent value setup). In general, bidders could be observing signals about others' valuations except in the Myersonian environment where we assume that bidders observe only their valuation. Fourth, the incumbents and the entrants who are allowed to participate are playing the auction game. If each bidder has a weakly dominant strategy (such as bidding one own's valuation in the Vickrey auction), the auction is referred to as a *dominant-strategy auction* and we then assume that buyers use their weakly dominant strategy. According to this terminology, second-price auctions with bid subsidies but also posted prices are dominant strategy auctions.

In such a case, our analysis is insensitive to the extra information about others' valuations bidders may receive after entry. We will be more explicit about the needed informational assumptions at the auction stage (for example about the number and identities of other participants) when considering formats in which there is no weakly dominant strategy as the first-price auction.

A key aspect of our analysis concerns the impact of (I, E) on the participation decisions. Due to our assumption that there is a large number of potential entrants in each group, the realized

¹⁵In this case we drop the variable z from our notation. Alternatively, we can consider the case where z would be common knowledge among bidders after their entry decisions. What we need when invoking the "independence" restriction is that we are in a setup in which, at the auction stage, the payoffs can be expressed as a function of the "assignment rule" as in Myerson (1981).

number of participants from each group will be distributed according to some Poisson distribution. Making entry endogenous will require that if the participation rate is positive in a given group, then the ex ante utility that an entrant from the group derives from participating should match his entry cost. That is, we assume that the probability that there are n_k entrants from group k for every $k \in \mathcal{E}$ is equal to $e^{-\sum_{k=1}^K \mu_k^*} \cdot \prod_{k=1}^K \frac{[\mu_k^*]^{n_k}}{n_k!}$ where the parameters μ_k^* , $k \in \mathcal{E}$, are determined in equilibrium so that $\mu_k^* > 0$ (resp. $\mu_k^* = 0$) implies that the expected payoff of an entrant from group k should be equal to (resp. be lower than) his entry cost C_k .

Assuming entry follows a Poisson distribution simplifies the exposition but is not essential for our results. As in Jehiel and Lamy (2015), we could have considered instead a setup with a large but finite number of potential entrants in each group.¹⁶ What matters for our results is that entrants get an expected payoff that is independent of the set-asides policy, which is so in a symmetric equilibrium of the game with finite but large enough number of potential entrants because the large number implies that entry with probability 1 cannot be part of an equilibrium (and thus the expected payoff of an entrant is invariably set at the entry cost). It should be mentioned that empirical works such as Athey, Levin and Seira (2011) or Athey, Coey and Levin (2013) that develop structural estimations of such models with endogenous entry obtain an average number of entrants (small firms according to their terminology) below three in the so-called “large sales” where large firms are incumbents.¹⁷ This suggests that the number of potential entrants need not be very large to have that the symmetric equilibrium is in mixed strategies. Furthermore, Athey, Coey and Levin (2013) also argue that pure-strategy equilibria would poorly fit the data.

2.2 Simple illustrations of the exclusion principle

To get some ideas about the potential effect of set-asides policies, we develop a few examples in which we consider the effect of excluding the incumbents.

Example 1a (the Vickrey auction with a single incumbent) Consider that 1) $X_S = 0$, 2) there is a single incumbent whose valuation is denoted by $x_I > 0$, 3) potential entrants are symmetric and all have the same valuation denoted by $x_E > 0$, and 4) the auction format is the Vickrey auction. In particular, when at least two entrants participate in the auction, their payoffs are null. We also assume that $E[x_E] > C$ in order to guarantee that the problem is not degenerate (otherwise entry would always be too costly). We do not make any additional restriction on the joint distribution of (x_E, x_I) . At the time they decide whether or not to enter the auction, we assume that potential entrants have no information about the realization of (x_E, x_I) .

Let μ_{in}^* (resp. μ_{out}^*) denote the equilibrium entry rate when the incumbent is not excluded (resp. is excluded). Note that if the entry rate is μ , the probability to be the sole entrant in the

¹⁶The Poisson distribution corresponds to the limit distribution of the number of entrants of each group of a model with a finite number of buyers per group taking independent decisions and as the number of buyers in each group goes to infinity (and assuming every individual entrant of a given group follows the same participation strategy). Jehiel and Lamy (2015) show that the equilibria in the Poisson specification correspond to the limit of equilibria as the number of potential entrants now assumed to be finite goes to infinity.

¹⁷Those large sales represent 80% of their timber auction sample. The average number of incumbents is 1.66.

auction is $e^{-\mu} > 0$. The equilibrium condition requiring that the expected payoff of an entrant in the auction is equal to his entry cost leads to:

$$e^{-\mu_{in}^*} \cdot E[\max\{x_E - x_I, 0\}] = C \quad (1)$$

if $E[\max\{x_E - x_I, 0\}] > C$ and $\mu_{in}^* = 0$ otherwise, and¹⁸

$$e^{-\mu_{out}^*} \cdot E[x_E] = C. \quad (2)$$

When the incumbent is not excluded, the expected revenue for the seller, denoted by $R_{with-1-Inc}$, is then

$$e^{-\mu_{in}^*} \cdot 0 + \mu_{in}^* e^{-\mu_{in}^*} \cdot E[\min\{x_E, x_I\}] + (1 - e^{-\mu_{in}^*} - \mu_{in}^* e^{-\mu_{in}^*}) \cdot E[x_E] \quad (3)$$

where the first term corresponds to the case without any entrant, the second term to the case with a single entrant and the third term to the case with at least two entrants. Plugging (1) into (3) we get the alternative expression

$$R_{with-1-Inc} = E[x_E] - \left(\frac{E[x_E]}{E[\max\{x_E - x_I, 0\}]} + \mu_{in}^* \right) \cdot C \quad (4)$$

if $\mu_{in}^* > 0$ (otherwise we have $R_{with-1-Inc} = 0$).

When the incumbent is excluded, the expected revenue of the seller, denoted by $R_{without-Inc}$, is then $(1 - e^{-\mu_{out}^*} - \mu_{out}^* e^{-\mu_{out}^*}) \cdot E[x_E]$ and plugging (2) into this expression, we obtain that

$$R_{without-Inc} = E[x_E] - (1 + \mu_{out}^*) \cdot C. \quad (5)$$

From (1) and (2), we have $e^{\mu_{out}^* - \mu_{in}^*} = \frac{E[x_E]}{E[\max\{x_E - x_I, 0\}]} > 1$. Since $e^x > 1 + x$ if $x > 0$, we obtain finally that $1 + \mu_{out}^* - \mu_{in}^* < \frac{E[x_E]}{E[\max\{x_E - x_I, 0\}]}$ or equivalently that

$$R_{without-Inc} > R_{with-1-Inc}. \quad (6)$$

Interestingly, this inequality does not depend on the relative strengths of the entrants and the incumbent. We will show that the benefit of excluding a single incumbent is actually very general and goes beyond this simple framework with ex-post symmetric entrants. \diamond

In the next example, we illustrate that the advantage of excluding the incumbent is not unique to the Vickrey format and that it applies to the first-price auction case in which the incumbent would have a right to match the best offer of the other participating bidders.

Example 1b (the first price auction with a right-to-match for the incumbent)

Consider again Example 1 but replace the Vickrey auction with the first-price auction allowing the incumbent to submit his bid after observing the bids of the entrants, assuming ties are broken

¹⁸The assumption $E[x_E] > C$ guarantees that a solution exists.

in favor of the incumbent. This gives an obvious second-mover advantage to the incumbent: In equilibrium, the incumbent bids zero if he knows he is the only bidder in the auction, and he matches the highest bid of the entrants if his own valuation is higher. Assume also, for simplicity, that the set of participants in the auction is publicly observed before bidders submit their bids so that entrants bid x_E if they know they face another entrant (in addition to the incumbent). Let $\beta(x_E)$ denote the equilibrium bid of an entrant when he knows there is no other entrant and his only competitor is the incumbent. We have naturally $\beta(x_E) \leq x_E$ (since any bid above x_E is weakly dominated). The equilibrium entry condition can be written as

$$e^{-\mu_{in}^*} \cdot E[(x_E - \beta(x_E)) \cdot 1[x_I < \beta(x_E)]] = C \quad (7)$$

if $E[(x_E - \beta(x_E)) \cdot 1[x_I < \beta(x_E)]] > C$ and $\mu_{in}^* = 0$ otherwise.¹⁹ A similar calculation leads to the analog of (4) for the seller's expected revenue when $\mu_{in}^* > 0$:

$$R_{with-1-Inc} = E[x_E] - \left(\frac{E[x_E]}{E[(x_E - \beta(x_E)) \cdot 1[x_I < \beta(x_E)]]} + \mu_{in}^* \cdot \gamma \right) \cdot C \quad (8)$$

where $\gamma = \frac{E[x_E - \beta(x_E)]}{E[(x_E - \beta(x_E)) \cdot 1[x_I < \beta(x_E)]]} \geq 1$. Since $\gamma \geq 1$, the previous argument leading to (6) holds a fortiori. Overall, we obtain that the benefit of excluding the incumbent is somehow reinforced in this auction. Intuitively, a mechanism that advantages the incumbent, or equivalently disadvantages the entrants, reduces the attractiveness for potential entrants (here formally we have $E[(x_E - \beta(x_E)) \cdot 1[x_I < \beta(x_E)]] \leq E[\max\{x_E - x_I, 0\}]$) but also increases the rents of the incumbent for any given number of entrants (here the expected rents of the incumbent is $E[(x_I - \beta(x_E)) \cdot 1[x_I \geq \beta(x_E)]]$ which is larger than the corresponding rate $E[(x_I - x_E) \cdot 1[x_I \geq x_E]]$ in the Vickrey auction).²⁰ \diamond

The next two examples explore the effect of set-asides in the presence of multiple incumbents. For this, we consider again the Vickrey auction.

Example 1c (the Vickrey auction with multiple symmetric incumbents) Consider again Example 1a but with two (or more) incumbents who have the same valuation ex-post. From the potential entrants' perspective, one or several incumbents makes no difference so that the equilibrium entry rate is equal to the same rate μ_{in}^* as in the case with a single incumbent. The expression of the seller's revenue, denoted by $R_{with-2-Inc}$, is then

$$R_{with-2-Inc} = e^{-\mu_{in}^*} \cdot E[x_I] + \mu_{in}^* e^{-\mu_{in}^*} \cdot E[x_I] + (1 - e^{-\mu_{in}^*} - \mu_{in}^* e^{-\mu_{in}^*}) \cdot E[\max\{x_E, x_I\}]. \quad (9)$$

After some calculation and plugging (1), we obtain that

$$R_{with-2-Inc} = E[\max\{x_E, x_I\}] - (1 + \mu_{in}^*) \cdot C \quad (10)$$

¹⁹We use the notation $1[A]$ where $1[A] = 1$ (resp. $1[A] = 0$) if statement A is true (resp. false).

²⁰An interesting feature here is that we can conclude that the exclusion of the incumbent is good even without solving the equilibrium strategy of the entrants.

if $\mu_{in}^* > 0$, and $R_{with-2-Inc} = E[x_I] \geq E[\max\{x_E, x_I\}] - C$ otherwise. Since $\mu_{in}^* < \mu_{out}^*$, we obtain that

$$R_{with-2-Inc} > R_{without-Inc}. \quad (11)$$

While it was beneficial to exclude the incumbent when there was only one, we see that exclusion may reduce revenues when there are multiple incumbents.

However, contrary to the result with a single incumbent, the impact of excluding the incumbents is ambiguous in general when there are multiple incumbents. In particular, the inequality (11) relies crucially on the implicit assumption that incumbents have no informational rents since their valuation are perfectly correlated. \diamond

Example 1d (the Vickrey auction with asymmetric incumbents) To deepen the discussion with multiple incumbents, consider a case with only two incumbents. We now introduce some asymmetry between the two incumbents: one incumbent, designated as a weak incumbent, has the valuation x_I with probability ϵ and a null valuation with the remaining probability $1 - \epsilon$, the other -strong- incumbent has valuation x_I . It is straightforward to see that excluding the weak incumbent is always detrimental since his presence or absence does not modify the entry rate while he may reduce the rent of the other incumbent. Whether it is profitable to exclude the strong incumbent and/or both incumbents depends on the strength of the weak incumbent: the weaker he is, the closer we are to a situation as if there were only one incumbent so that exclusion is profitable. To illustrate how rich the situation can be with multiple incumbents, we provide an example in the Appendix such that it is detrimental to exclude each incumbent in isolation but it would be good to exclude both incumbents. \diamond

3 Preliminaries

3.1 Some definitions

In this section, we assume that the auction format is the Vickrey auction. We let $N = (n_1, \dots, n_K) \in \mathbb{N}^K$ denote a realization of the profile of entrants, $N_{-k} = (n_1, \dots, n_{k-1}, n_k - 1, n_{k+1}, \dots, n_K)$ and $N_{+k} = (n_1, \dots, n_{k-1}, n_k + 1, n_{k+1}, \dots, n_K)$. For a given (nonempty) set of incumbents $I \subseteq \mathcal{I}$ and $i \in I$, we let $I_{-i} = I \setminus \{i\}$. The following notation will be useful to present the analysis.

- $F^{(1:N \cup I)}(x) := E_Z[\prod_{k=1}^K [F_k(x|Z)]^{n_k} \cdot \prod_{i \in I} F_i^I(x|Z)]$ denotes the CDF of the first order statistic of bidders' valuations among the set N of entrants and the set $I \subseteq \mathcal{I}$ of incumbents. If $N = (0, \dots, 0)$ and $I = \emptyset$, then we adopt the convention that $F^{(1:N \cup I)}(x) = 1$.
- $P(N|\mu) = e^{-\sum_{k=1}^K \mu_k} \cdot \prod_{k=1}^K \frac{\mu_k^{n_k}}{n_k!}$ denotes the probability of the realization N when the entry rate vector is μ , namely when the Poisson distribution of group k buyer has mean

$\mu_k \geq 0$ for every $k \in \mathcal{E}$.

- $V_k^{ent}(N, I)$ [resp. $V_i^{inc}(N, I)$] denotes the expected (interim) gross payoff (i.e. without taking into account the entry costs) of a buyer from group k [resp. the incumbent $i \in I$] when the set of participants consists of the profile of potential entrants $N \in \mathbb{N}^K$ with $n_k \geq 1$ [resp. $N \in \mathbb{N}^K$] and the set of incumbents $I \subseteq \mathcal{I}$.
- $\Pi_k^{ent}(\mu, I) = \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot V_k^{ent}(N, I) - C_k$ denotes the expected (ex ante) payoff of a group k buyer net of the entry cost C_k when the profile of entry rate is $\mu \in \mathbb{R}_+^K$ and the set of incumbents is $I \subseteq \mathcal{I}$.²¹
- $\Pi_i^{inc}(\mu, I) = \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot V_i^{inc}(N, I)$ denotes the expected (ex ante) payoff of the incumbent $i \in I$ when the profile of entry rate is $\mu \in \mathbb{R}_+^K$ and the set of incumbents is $I \subseteq \mathcal{I}$.

For every realization (N, I) of participants, we define the expected (interim) gross welfare (i.e. the sum of all agents' utilities excluding the entry costs) by

$$W(N, I) := X_S \cdot F^{(1:N \cup I)}(X_S) + \int_{X_S}^{\bar{x}} x dF^{(1:N \cup I)}(x)$$

and the expected (interim) seller's payoff (consisting of X_S when the good is not sold and the revenue otherwise) by

$$\Phi(N, I) := W(N, I) - \sum_{k=1}^K n_k \cdot V_k^{ent}(N, I) - \sum_{i \in I} V_i^{inc}(N, I).$$

With some abuse of terminology, we refer to $\Phi(N, I)$ as revenue in the rest of the paper.

We can now state more formally the conditions for a profile of entry rates μ to be an equilibrium. For a given set-asides policy (I, E) , we say that an entry profile μ is part of an equilibrium if $\mu_k = 0$ for $k \notin E$, and for any $k \in E$,

$$\Pi_k^{ent}(\mu, I) \underset{(resp. \leq)}{=} 0 \text{ if } \mu_k \underset{(resp. =)}{>} 0. \quad (12)$$

Let $J(I, E) \subseteq \mathbb{R}_+^K$ denote the set of entry profiles compatible with equilibrium behavior for the set-asides policy (I, E) , and let $\mu^*(I, E) \in J(I, E)$ refer to one such equilibrium.²²

For a given participation profile μ and a set of incumbents $I \subseteq \mathcal{I}$, we define the expected (ex ante) total net welfare (net of the expected entry costs) by

²¹Note that from the perspective of any entrant no matter what his group k is, the probability that he faces the set of entrants N (excluding himself) is also equal to $P(N|\mu)$. This fundamental property of Poisson games is referred to as environmental equivalence in Myerson (1998).

²²We will establish in the proof of Lemma 3.1 that $J(I, E) \neq \emptyset$ for any pair (I, E) so that such an equilibrium exists. The participation rates cannot be infinite as it would result in negative expected net payoffs when participating (since the entry costs C_k are assumed to be strictly positive).

$$TW(\mu, I) := \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot W(N, I) - \sum_{k=1}^K \mu_k \cdot C_k \quad (13)$$

and the corresponding expected revenue by $R(\mu, I) := \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot \Phi(N, I)$. From the equilibrium condition (12), the expected revenue of the seller given the set-asides policy (I, E) can be rewritten as

$$R(\mu^*(I, E), I) = TW(\mu^*(I, E), I) - \sum_{i \in I} \Pi_i^{inc}(\mu^*(I, E), I). \quad (14)$$

We are interested in how $TW(\mu^*(I, E), I)$ and $R(\mu^*(I, E), I)$ vary with the set-asides policy (I, E) . Clearly, $TW(\mu, I)$ and $R(\mu, I)$ are increasing in the set of incumbents I for a given μ because, in the Vickrey auction, the good is allocated to the agent who values the good most and the payment when the good is sold can only increase when there are more participants. Forbidding the access of some incumbents would a priori boost the participation rate of the potential entrants, which would be favorable both to TW and R . The question is how the two effects aggregate.

3.2 Welfare criterion

When the criterion is welfare, we show that it is never good to exclude incumbents or entrants. The following lemma is a key property used repeatedly in our analysis. For any set I of allowed incumbents, the equilibrium entry profile must be one that maximizes the welfare given I . While Jehiel and Lamy (2015) establish this result in environments without incumbents, the extension to the case with incumbents is straightforward.²³

Lemma 3.1 $\mu^*(I, \mathcal{E}) \in \text{Arg max}_{\mu \in \mathbb{R}_+^K} TW(\mu, I) \neq \emptyset$ for any $I \subseteq \mathcal{I}$.

To help understand some of the following results, we now provide a sketch of the key steps in the proof (while for completeness a full proof appears in the Appendix). A fundamental property of the Vickrey auction is that the ex post utility of each bidder corresponds exactly to his marginal contribution to the welfare.²⁴ Applied to a potential entrant from group k , this implies (from an interim perspective) that

$$W(N_{+k}, I) - W(N, I) = V_k^{ent}(N_{+k}, I). \quad (15)$$

From an ex ante perspective and given the Poisson model, such a property translates into:

$$\frac{\partial TW(\mu, I)}{\partial \mu_k} = \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot V_k^{ent}(N_{+k}, I) - C_k = \Pi_k^{ent}(\mu, I). \quad (16)$$

²³The result holds true for all sets of incumbents because from the viewpoint of entrants as well as from the welfare viewpoint, the presence of incumbents is equivalent to that of a seller with a stochastic reservation value (see Lamy (2013) for the analysis of the Vickrey auction with endogenous entry when the reserve price is randomly determined).

²⁴The first work on auctions with entry having stressed this property is Engelbrecht-Wiggans (1993).

That is, the entry profiles resulting from equilibrium behavior correspond to local maxima of the total welfare function. The final step in the proof consists in showing that the function $\mu \rightarrow TW(\mu, I)$ is globally concave, thereby ensuring that any local maximum is a global maximum.²⁵

Since $TW(\mu, I)$ is non-decreasing in I , it follows that $\max_{\mu \in \mathbb{R}_+^K} TW(\mu, I)$ is also non-decreasing in I . From Lemma 3.1, we obtain then that excluding incumbents can only reduce the total welfare. Hence, the first-best welfare $\max_{\mu \in \mathbb{R}_+^K} TW(\mu, \mathcal{I})$ can be implemented with the no-exclusion policy.

Proposition 3.2 *In the Vickrey auction, the welfare-optimal set-asides policy involves no exclusion.*

While this subsection is concerned with welfare, we make the following three basic observations concerning revenues. If there are no incumbents, the total net welfare and the seller’s revenue coincide as shown in (14). As a corollary of Proposition 3.2, we have that

Corollary 3.3 *In the Vickrey auction without incumbents, the revenue-optimal set-asides policy involves no exclusion.*²⁶

We refer to cases in which the rents of incumbents are null, i.e. $V_i^{inc}(N, \mathcal{I}) = 0$ for any $N \in \mathbb{N}^K$ and $i \in \mathcal{I}$, as situations with “full competition among incumbents”. Obviously, several incumbents are required for this to be true and it typically arises as illustrated in Example 1c when there are several incumbents with the same highest valuations. In the full competition among incumbents case, we get from (14) that $\max_{\mu \in \mathbb{R}_+^K} TW(\mu, \mathcal{I})$ is an upper bound on the seller’s expected revenue. Since Proposition 3.2 has established that this bound is reached by the seller under the no exclusion policy, we can extend an observation made in Example 1c:

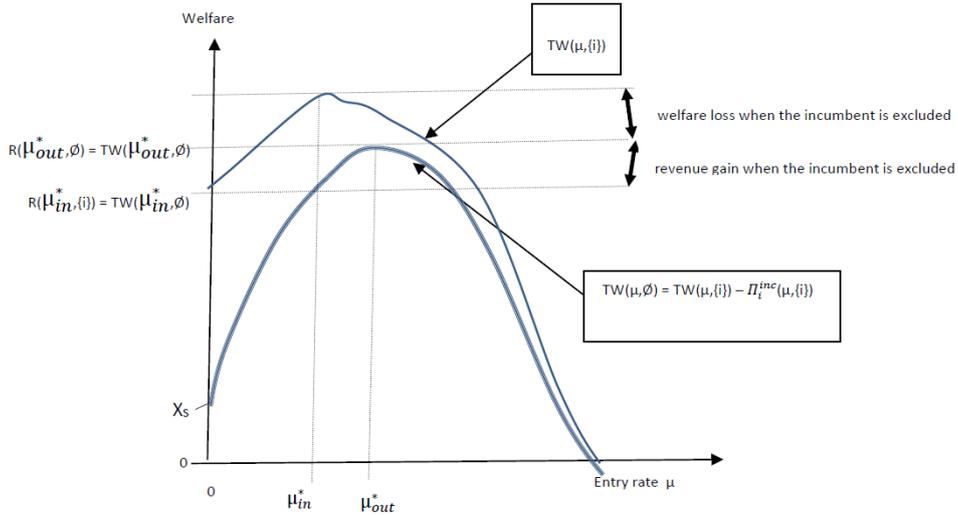
Corollary 3.4 *In the Vickrey auction with “full competition among incumbents”, the revenue-optimal set-asides policy involves no exclusion.*

When there is a single incumbent, it may be the case that the incumbent is an organization owned by the seller (e.g. another public administration in public procurements) in which case the designer’s objective corresponds to the seller’s revenue augmented by the payoff of the incumbent. This case is next referred to as the case where the seller internalizes the incumbent’s payoff. The seller’s objective then coincides with welfare in equilibrium given that entrants’ payoffs are determined by their participation costs. In the Vickrey auction, the revenue-optimal set-asides policy thus involves no exclusion in this case, as in the case without incumbents.

²⁵The entry game presents some analogy with the theory of potential games (Monderer and Shapley, 1996) and we know from it that any global maximum of the potential function constitutes an equilibrium and that the converse is true if the potential function is concave (Neyman, 1997). The game where a finite set of players makes (simultaneous) pre-participation investments before bidding in a Vickrey auction corresponds precisely to a potential game where the potential function is the welfare. A similar observation (implicitly) appears in Bergemann and Välimäki (2002) who derive from it the existence of an efficient equilibrium both from an ex ante and an ex post perspective. Technically, there is still a difference in our environment since the Poisson model involves implicitly a continuum of players.

²⁶Jehiel and Lamy (2015) adopt a mechanism design approach à la Myerson (1981) and show the much stronger result that the Vickrey auction is the optimal mechanism when there are no incumbents.

Figure 1: Illustration of Theorem 1 when $K = 1$



4 The Vickrey auction with a single incumbent

4.1 Excluding a single incumbent boosts revenues

In the presence of one or several incumbents, revenue maximization involves a trade-off between welfare maximization and the minimization of the rents of the incumbents (as reflected by (14)). On the one hand, from Proposition 3.2, set-asides impact negatively the revenue through the welfare term. On the other hand, excluding incumbents can serve the purpose of eliminating the rents of these. Although it is in general unclear what the direction of the trade-off is, it turns out that when there is a single incumbent, excluding the incumbent is always good for revenues in the Vickrey auction.

In words, the argument is as follows. In the Vickrey auction, for any realization of the profile of entrants, the expected rent of the incumbent coincides exactly with his marginal contribution to the welfare (this is the fundamental property of the Vickrey auction already used for entrants to prove Lemma 3.1). We obtain then (from eq. (14)) that the expected revenue of the seller coincides with the (hypothetical) total welfare were the incumbent to be absent. Since the latter welfare is maximized when the incumbent is excluded and all entrants -whatever their group- are allowed to participate, we obtain:

Theorem 1 *When there is a single incumbent, the revenue-optimal set-asides policy in the Vickrey auction consists in excluding the incumbent and allowing entrants whatever their group to participate. That is, the revenue-maximizing set-asides policy is $(I, E) = (\emptyset, \mathcal{E})$.*

When there is a single group of entrants, the following proof is illustrated in Figure 1. The revenue gain for exclusion comes from the gap between $\mu^*(\emptyset, E) \equiv \mu_{out}^*$ and $\mu^*({i}, E) \equiv \mu_{in}^*$.

Proof Since the ex post utility of the incumbent coincides with his marginal contribution to the welfare, we obtain the analog of (15) but for an incumbent $W(N, I) - W(N, I_{-i}) = V_i^{inc}(N, I)$ for any realization N of the profile of entrants, and then from an ex ante perspective

$$\Pi_i^{inc}(\mu, I) = TW(\mu, I) - TW(\mu, I_{-i}) \quad (17)$$

which can be viewed as the analog of (16). Plugged into (14) and for environment with a single incumbent, we obtain that for any set-asides policy (I, E) ,²⁷ that the revenue of the seller is given by

$$R(\mu^*(I, E), I) = TW(\mu^*(I, E), \emptyset) \leq TW(\mu^*(\emptyset, E), \emptyset) = R(\mu^*(\emptyset, E), \emptyset) \quad (18)$$

where the middle inequality comes from Lemma 3.1. **Q.E.D.**

4.2 When the incumbent must be in

In some cases, it may be that the seller is not allowed to exclude the incumbent. One may then be interested in whether or not it is good for revenues to exclude some groups of entrants from the auction. Obviously, if there is only one group of entrant, this cannot be good as excluding the entrants would reduce the revenues to X_S (and the seller is bound to get more than X_S in the Vickrey auction with positive participation). When there are several groups of entrants, one might have thought based on Theorem 1 that it is not a good idea to exclude any group of entrants. This turns out to be incorrect as illustrated in the following example.

Example 2 Consider one incumbent having for sure valuation x_I . Consider two groups of potential entrants. With probability $q \in (0, 1)$, all entrants have the high valuation $\bar{x}_E > x_I$. With probability $1 - q$, entrants from group 1 (resp. 2) have the low valuation $\underline{x}_E^1 < x_I$ (resp. $\underline{x}_E^2 < x_I$). We assume that $\underline{x}_E^1 < \underline{x}_E^2$ and that their entry costs, denoted by C_k for $k = 1, 2$, are such that $C_1 < C_2$ with the difference $C_2 - C_1$ being small enough as made precise below. We also assume that $\bar{x}_E - x_I > C_2$, which guarantees that some entry arises in equilibrium. The expected gross profit of an entrant is the same whether he comes from group 1 or group 2. However, since the entry cost is bigger in group 2 than in group 1, when there are no set-asides, then only bidders from group 1 participate. Formally, the profile of equilibrium entry rates (μ_1, μ_2) without set-asides is such that

$$q \cdot e^{-\mu_1}(\bar{x}_E - x_I) = C_1$$

and $\mu_2 = 0$. By contrast, when bidders from group 1 are excluded then the equilibrium entry rate of group 2 is given by $\tilde{\mu}_2$ such that

$$q \cdot e^{-\tilde{\mu}_2}(\bar{x}_E - x_I) = C_2.$$

²⁷Namely, either if $I = \{i\}$ or if $I = \emptyset$.

As the difference $C_2 - C_1$ gets small, we have that $\tilde{\mu}_2 \approx \mu_1$. The welfare is smaller with exclusion (as we already know from Proposition 3.2), but the difference becomes negligible here. However, given that $\underline{x}_E^1 < \underline{x}_E^2$, the seller's revenue is larger when group 1 is excluded (and the difference, which is approximately equal to $(1 - e^{-\mu_1}) \cdot (1 - q) \cdot (\underline{x}_E^2 - \underline{x}_E^1) > 0$, is non negligible when $C_2 - C_1$ gets small). \diamond

In the above example, there is a discrepancy between the effect of an entrant on the reduction of the rents of the incumbent and his effect on the increase of the welfare. Group 2's bidders are more effective for the former while group 1's bidders are more effective for the latter. It is then intuitive that it can be profitable to exclude those potential participants who have a greater ability to increase the welfare (so that they will enter without exclusion and discourage the other group to participate) but less ability to reduce the incumbent's rents.

Understanding further which groups of entrants should be excluded when the incumbent is present is not straightforward given the potential complex effect of E on the participation rates. To illustrate some counter-intuitive effects, Example 3 in Appendix shows that it may be good to exclude a group that is dominated both in terms of entry costs and in terms of the distribution of valuations by another group.²⁸

4.3 Simple extensions of the exclusion principle

Several remarks in relation to the exclusion principle are in order. First, for any general assignment problem and private value environments with quasi-linear utilities, the exclusion principle of Theorem 1 extends. The mechanism should be the pivot mechanism that requires efficient assignments and that participants pay the welfare loss their presence imposes on others. If there are no incumbents, the designer can be shown to achieve the highest possible revenue in the pivot mechanism if we assume she can coordinate entry on the equilibrium she likes best (see Jehiel and Lamy (forthcoming)). As a corollary, the exclusion principle of Theorem 1 follows for exactly the same reasons as in the one-object case given that an economic agent who participates for sure (the incumbent) would still grab his marginal contribution to the welfare in the pivot mechanism. Such an observation applies to multi-object generalized Vickrey auction as discussed in greater details in Section 7.2. It applies also to settings in which there would be costs attached to the processing of bids in which case the pivot mechanism would include a fee equal to the processing cost on the top of the second-price auction with a reserve price equal to X_S .²⁹

²⁸The intuition is the following: The example relies on a "strong group" where valuations among bidders are perfectly correlated and a "weak group" where valuations are drawn independently of each other. Due to the perfect correlation in the strong group, bidders do not enter much in equilibrium (because their payoff are null when two of them enter). By contrast, the equilibrium level is higher in the weak group (which is useful to reduce the rents of the incumbent when the latter wins the auction but has to pay the highest valuation among his competitors) which counterbalances at the end the fact that for a given set of competitors, bidders from the strong group increase more the revenue than those from the weak group.

²⁹We note that if entry fees can not be used, processing costs would provide a direct motive for set-asides, even when participation is exogenous. In a related vein, and in contrast with Myerson (1981), Crémer, Spiegel and Zheng (2007) show in a dynamic mechanism framework with exogenous entry but processing costs that set-asides could be beneficial. Their result relies on the possibility to solicit bids in a sequential way, a channel for exclusion

Second, instead of considering infinite populations of entrants in each group (this has led us to adopt the Poisson formulation), the exclusion result of Theorem 1 would similarly hold in a model with a finite number of potential entrants in each group as considered in Jehiel and Lamy (2015) provided that (i) each group is sufficiently large so that entry with probability 1 is not an equilibrium whatever the group in the seller's most preferred (symmetric) equilibrium, (ii) different entrants from the same group participate with the same probability (symmetry assumption) and (iii) the seller is able to guarantee that the profile of equilibrium entry probabilities, when the incumbent is out, is the one she prefers (among equilibrium ones).³⁰

Third, our exclusion principle is robust to the introduction of ex-ante heterogeneities between bidders of the same group (e.g. if entry costs differ across bidders as in Krasnokutskaya and Seim (2011)) provided that the rents those heterogeneities induce are limited. Formally and as argued in Jehiel and Lamy (2015), if the sum of those rents are bounded by ϵ in an environment without incumbents, then the revenue gain from switching from the Vickrey auction to the optimal mechanism is bounded by ϵ . This further implies that excluding the incumbent induces at worst a revenue loss of ϵ .

Fourth, assuming the seller cannot use reserve prices, Theorem 1 extends straightforwardly to the extent that the incumbent's valuation is always positive. In the second price auction without reserve price, the seller's revenue corresponds to her revenue as if her true valuation were 0 augmented by the probability that the good remains unsold times X_S . As a direct application of Theorem 1, the first term is maximized when the incumbent is excluded. We conclude after noting that the second term is also maximized when the incumbent is excluded given that when the incumbent's valuation is positive, the good is always sold when he is in.³¹

5 Beyond the Vickrey auction

In some procurement applications, bidders are not treated symmetrically. For example, some bidders may be better at renegotiating the terms of the contracts, thereby giving them an advantage at the auction stage (winning at the same nominal price would translate in a lower effective price for such bidders). Or bidders may be asymmetric in the risk of breakdown. In other cases, some bidders may be allowed to bid after seeing the offers of others, thereby giving them a second-mover advantage. In all such cases in which bidders may be treated asymmetrically, the auction format cannot be viewed as being equivalent to the Vickrey auction, and it is of interest to analyze whether and when excluding the incumbent or some groups of entrants may be beneficial to the seller.

that is absent in our model.

³⁰This would also correspond to an equilibrium selection that is popular in game theory when one deals with a potential game (as it is the case here and where the potential function is the welfare net of the entry costs, see footnote 25): it consists of selecting the equilibrium that maximizes globally (and not only locally) the potential (see Haufbauer and Sorger (1999) and Carbonell-Nicolau and McLean (2014)).

³¹More generally, the argument extends as long as the reserve price is below the seller's valuation while the incumbent's valuation is always above it.

Our first insight in this Section is derived from the following observation. A simple inspection of the argument used to prove Theorem 1 reveals that the benefit of excluding the incumbent carries over to situations in which the rent obtained by the incumbent is no smaller than his marginal contribution to the welfare (in the Vickrey auction, it is just equal) and the good would be allocated efficiently among the entrants when the incumbent is out. We identify classes of auction formats in which this would be the case so that excluding the incumbent is revenue-enhancing: Such classes roughly correspond to formats in which the incumbent always gets the good when he values it most. In a related vein, we establish that when there is a single group of entrants and the auction format is such that the incumbent never gets the good when he does not value it most while an entrant always gets it when he values it most, then excluding the incumbent is detrimental when the seller internalizes the incumbent's payoff. In order to cover also the exclusion of groups of entrants, we also consider mechanisms that over-assign the good to the entrants from a first group and under-assign the good to the entrants from the second group, and we show that it is never good to exclude the entrants from this second group.

After deriving our results within an abstract class of mechanisms in which bidders have a (weakly) dominant strategy, we show how these results can be used in more concrete applications. We also discuss the implications of our results beyond dominant-strategy auctions, in particular when first-price auctions are used.

5.1 Definitions

An assignment rule, denoted by ϕ , is a function which maps any realization of bidders' valuations to a vector of probabilities characterizing the probability that each participant receives the good. We say that an assignment rule is deterministic if the vectors of probabilities are composed only of 0 and 1. E.g. the assignment rule associated to the Vickrey auction is one that assigns probability one to the participant (including the seller herself) who has the highest valuation.³²

Definition 2 *For a given assignment rule, we say that the good is over-assigned (resp. under-assigned) to a bidder if this bidder gets the good with probability 1 [resp. 0] whenever (ex post) efficiency dictates to assign (resp. not to assign) it to him.*

Relatedly, we will say that an auction over-assigns (resp. under-assigns) the good to a given bidder if in the associated assignment rule (resulting from equilibrium behavior), the good is over-assigned (resp. under-assigned) to the given bidder. For auctions in which bidders have a dominant strategy, we will use the assignment rule ϕ induced by the auction mechanism to designate the auction format.³³

³²When several bidders have the same valuation in the Vickrey auction, it does not matter how to break ties in terms of bidders payoffs.

³³In general, this terminology is abusive because it relies on the endogenous equilibrium behavior induced by the auction rules.

Let us illustrate the previous definition with a class of assignment rules when there is at most one incumbent. Consider $r \in \mathbb{R}_+$ and $b : [r, \infty) \rightarrow [X_S, \infty)$ an increasing and continuous function with $b(r) = X_S$. Let us define an assignment rule $\phi(b, r)$ in the following way:³⁴

- When all entrants have a valuation below X_S and the incumbent has a valuation below r , the seller keeps the good,
- When the incumbent has a valuation below r and there is at least one entrant with a valuation strictly above X_S , the good is assigned to the entrant with the highest valuation,
- When the incumbent has a valuation strictly above r and there is no entrant with a valuation strictly above X_S , the good is assigned to the incumbent,
- When the incumbent has a valuation x_I strictly above r and the highest valuation among the entrants x_E is strictly above X_S , the good is assigned to the incumbent if $b(x_I) \geq x_E$ and to the entrant with the highest valuation otherwise.

It is readily verified that the good is over-assigned (resp. under-assigned) to the incumbent in a $\phi(r, b)$ -assignment rule with $b(x) \geq x$ and $r \leq X_S$ (resp. $b(x) \leq x$ and $r \geq X_S$). The assignment rule associated to the Vickrey auction is the knife-edge case where $r = X_S$ and $b(x) = x$.

This construction is illustrated in Figure 2 in which two assignment rules are depicted in bold. The assignment rule $\phi(r, b)$ delineates three areas: in the red rectangle the seller keeps the good, above the bold line associated to b and above the red rectangle the entrant with the highest valuation gets the good, below the bold line associated to b and on the right to the red rectangle the incumbent gets the good. This assignment rule is inefficient. More precisely, inefficiencies occur when we are in the shaded area: the incumbent gets the good although efficiency would dictate to put it in the hands of another agent (an entrant or the seller herself). In this assignment rule, the good is over-assigned (resp. under-assigned) to the incumbent (resp. the entrants).

The assignment rule $\phi^{Myerson}$ depicts the one associated to the optimal mechanism in a Myersonian environment with the regularity assumption that the function $x \rightarrow x - \frac{1-F^I(x)}{f^I(x)}$ is increasing, as characterized in Jehiel and Lamy (2015). The good is under-assigned to the incumbent and over-assigned to the entrants.³⁵

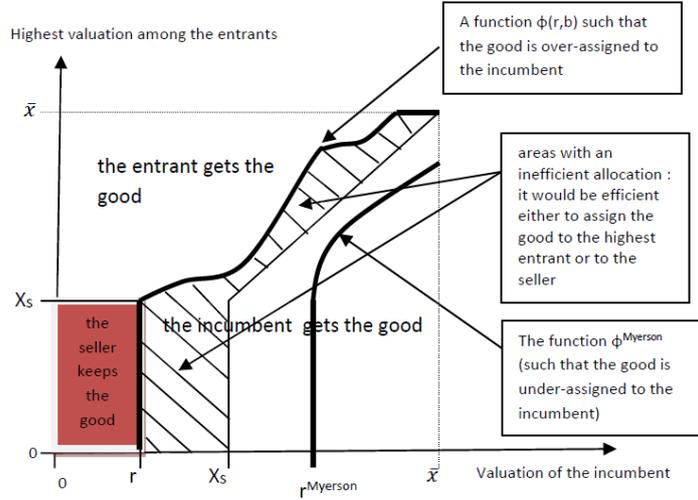
5.2 Some fundamental properties

It is well-known in mechanism design that bidders' expected rents are characterized (either for dominant-strategy mechanisms or for general mechanisms in Myersonian environments) by

³⁴If the incumbent is excluded then it corresponds to say that he has a null valuation so that he never gets the good. Similarly, if there is no entrants we adopt the convention that the highest valuation among the entrants is null.

³⁵More precisely, we have $\phi^{Myerson} = \phi(r^{Myerson}, b^{Myerson})$ where $r^{Myerson}$ denotes the solution to $r^{Myerson} - \frac{1-F^I(r^{Myerson})}{f^I(r^{Myerson})} = X_S$ and the function $b^{Myerson}(x) = x - \frac{1-F^I(x)}{f^I(x)} \leq x$ where F^I (resp. f^I) denotes the CDF (resp. PDF) of the incumbent's valuation.

Figure 2: Examples of $\phi(r, b)$ -assignment rules



the assignment rule and the expected monetary transfers of bidders with the lowest type. In the next results (up to Section 5.5), we consider so-called “dominant-strategy auctions without participation fees”, i.e. dominant-strategy auctions such that the transfer of a bidder with zero valuation is null.

When comparing the seller’s revenue in auction formats under exogenous participation, a crucial element is how bidders’ payoffs vary with the assignment rule. From standard arguments in mechanism design, it is well-known that the more a bidder gets the good the larger his payoff. Limiting bidders’ payoff pushes thus in favor of sometimes not assigning the good to a bidder although it would be efficient to do so. When comparing the seller’s revenue, e.g. with respect to the set-asides policy, under endogenous participation, what matters is no longer bidders’ payoffs per se (they are fixed exogenously for the entrants) but rather whether bidders’ payoffs are larger or smaller than their marginal contribution to the welfare. The aim of the next lemmas is to relate the difference between bidders’ rents and their marginal contribution to the welfare to the notions of over- or under-assigning the good to some bidders. Those results are intuitive and use standard mechanism design techniques. Nevertheless, it should be highlighted that as far as we know they do not appear in the previous literature and could potentially be useful beyond the present study of set-asides.

Lemma 5.1 *In a dominant-strategy auction without participation fees that over-assigns (resp. under-assigns) the good to a given incumbent i and that always assigns the good efficiently among the remaining bidders and the seller, the payoff of incumbent i is larger (resp. smaller) than his marginal contribution to the welfare.*

In other words, for any realization of bidders’ valuations, if the good is over-assigned (resp. under-assigned) to incumbent i , then he grabs more (resp. less) than the ex post surplus he

brings by his presence. Formally, after adapting our notation about the welfare and payoff functions to make them depend on the assignment rule ϕ associated to the given dominant-strategy auction, we obtain $W(N, I; \phi) - W(N, I_{-i}; \phi) \leq V_i^{inc}(N, I; \phi)$ (resp. $W(N, I; \phi) - W(N, I_{-i}; \phi) \geq V_i^{inc}(N, I; \phi)$) if the good is over-assigned (resp. under-assigned) to the incumbent $i \in I$ and if the auction always assigns the good efficiently among the remaining bidders and the seller.

From an ex ante perspective, when the good is over-assigned to incumbent $i \in I$, we obtain then for any entry profile μ that

$$TW(\mu, I; \phi) - TW(\mu, I_{-i}; \phi) \leq \Pi_i^{inc}(\mu, I; \phi). \quad (19)$$

As an illustration, assume there is a single incumbent (I is a singleton) and consider a dominant strategy auction without participation fees that implements a $\phi(b, r)$ -assignment rule. If we let $G_\mu(\cdot)$ denote the CDF of the highest valuation among the entrants when the entry profile is $\mu \in \mathbb{R}_+^K$, assuming the incumbent's valuation is distributed independently of the entrants' valuation, standard calculation leads to (see the Appendix for details)

$$\begin{aligned} TW(\mu, I; \phi(b, r)) - TW(\mu, \emptyset; \phi(b, r)) &= \Pi^{inc}(\mu, I; \phi(b, r)) + (r - X_S) \cdot G_\mu(X_S) \cdot (1 - F^I(r)) \\ &+ \int_r^\infty (x - b(x)) \cdot (1 - F^I(x)) \cdot d[G_\mu(b(x))]. \end{aligned} \quad (20)$$

It is clear from the previous expression that the more the good is assigned to the incumbent (when $b(x)$ gets larger and r smaller), the wider is the discrepancy between his payoff and his marginal contribution to the welfare.

As will prove useful when considering the exclusion of (some groups of) entrants, a similar argument can be developed to compare a given bidder's payoff to his marginal contribution to the welfare in the class of loser neutral assignment rules. In words, those rules require that if a given bidder does not win the auction (or equivalently is a loser), then the way the good is assigned among his opponents does not depend on his own report and is the same as if he were excluded from the auction.

Formally, letting $\phi_i(x_1, \dots, x_n)$ denote the probability that bidder i (with valuation x_i) gets the good in the assignment rule ϕ , loser neutrality is defined as:

Definition 3 *An assignment rule is loser neutral if for any vector $X := (x_1, \dots, x_n)$ and any bidder i such that $\phi_i(X) < 1$, we have $\phi_j(X) = (1 - \phi_i(X)) \cdot \phi_j(X_{-i})$ for any $j \neq i$ where*

$$X_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

We have:

Lemma 5.2 *In a loser neutral dominant-strategy auction without participation fees that over-assigns (resp. under-assigns) the good to a given bidder, the payoff of the given bidder is larger*

(resp. smaller) than his marginal contribution to the welfare.

5.3 Set-asides in dominant-strategy auctions

As an application of Lemma 5.1, the payoff of the incumbent is always larger than his marginal contribution to the welfare in auctions that over-assigns the good to the incumbent. This implies that the seller's revenue is still bounded from above by the total welfare maximizing solution in the absence of the incumbent (specifying that the good is allocated to the agent -entrant or seller- with highest valuation and participation rates are defined to maximize total expected welfare).³⁶ Since such an upper bound on revenues is reached by excluding the incumbent and keeping all groups of entrants, we obtain the following result, which constitutes a generalization of Theorem 1.

Theorem 2 *Consider an environment with a single incumbent and a dominant-strategy auction without participation fees where the good is over-assigned to the incumbent and that always assigns the good efficiently among the remaining bidders and the seller. The revenue-optimal set-asides policy consists in excluding the incumbent and allowing all groups of entrants to participate.*

When there is a single group of entrants, Theorem 2 is illustrated in Figure 3: We note first that in contrast to the Vickrey auction, excluding the incumbent may be welfare-improving (this is so for sure if $TW(\mu, \{i\}; \phi) < TW(\mu_{out}^*, \emptyset)$, for each $\mu \geq 0$) when the good is over-assigned to the incumbent. Note incidentally that for any kind of auctions that assign the good efficiently among entrants, if exclusion is welfare-improving, it is for sure revenue-improving. For the revenue, exclusion is even more profitable for two reasons: On the one hand, the equilibrium revenue with the incumbent i will stand on the curve $TW(., \{i\}; \phi) - \Pi_i^{inc}(., \{i\}; \phi)$ which is now below the curve $TW(., \emptyset)$. On the other hand, if the good is over-assigned to the incumbent the entry rate is reduced compared to the Vickrey auction: $\mu^*(\{i\}, \phi) < \mu^*(\{i\})$ and so we stand more on the left of the $TW(., \{i\}; \phi) - \Pi_i^{inc}(., \{i\}; \phi)$ curve.

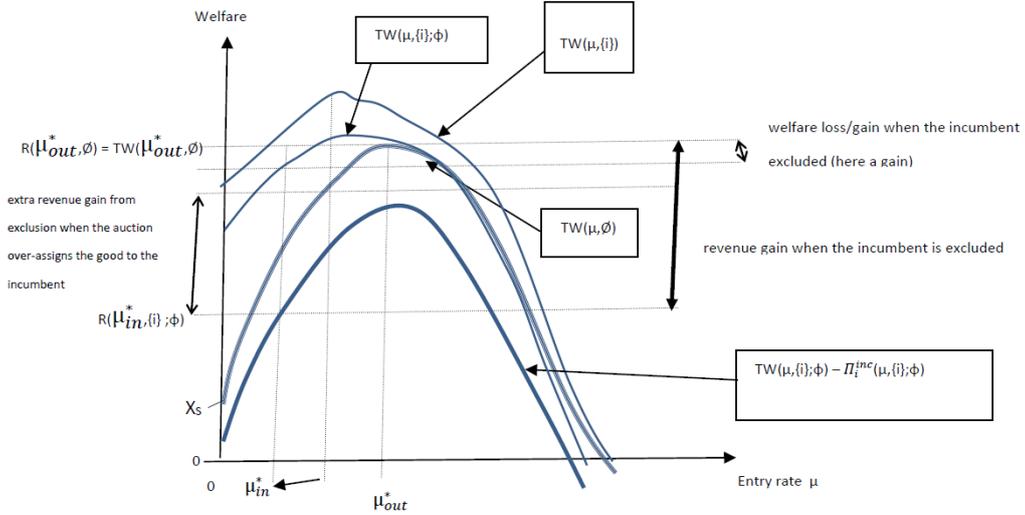
Assume now that the objective is the seller's revenue augmented with the incumbent's payoff. As a mirror of Theorem 2, if the assignment favors entrants with respect to the incumbent, it is intuitive that it would be optimal to keep a fortiori the incumbent.

Theorem 3 *Consider an environment with a single incumbent, a single group of entrants ($K = 1$) with $W(1, \emptyset) > X_S + C$ and a dominant-strategy auction without participation fees such that the good is under-assigned to the incumbent and over-assigned to the entrants and such that the probability to win the good for an entrant with a given signal weakly (resp. strictly) decreases when an extra entrant (resp. the incumbent) participates.³⁷ If the seller's objective is her revenue augmented with the incumbent's payoff, then the seller is strictly worst-off by excluding the incumbent.*

³⁶It is implicit throughout our analysis that $\mu^*(I, E; \phi)$ is well defined (namely that an equilibrium exists) for the class of auctions ϕ we consider. See Jehiel and Lamy (2015) for a proof of existence for a very general class of mechanisms.

³⁷This last assumption does not depend solely on the assignment rule, but also possibly on the underlying valuation distributions. In $\phi(r, b)$ -assignment rules, note that the probability to win the good can not increase with

Figure 3: Illustration of Theorem 2 when $K = 1$



The intuition for why excluding the incumbent lowers the welfare is as follows: since the good is over-assigned to the entrants, it must be that the entry rate is above the efficient one. Furthermore, excluding the incumbent will increase the entry rate and thus further moves away from the efficient one. Last given that the good is under-assigned to the incumbent, for any given entrant rate, the welfare with the incumbent is above the welfare if the incumbent were excluded.

Turning to entrants, we can show making use of Lemma 5.2 that in the absence of incumbents and with two groups of entrants, if the good is over-assigned to bidders in one group and under-assigned to the bidders in the other group, then it is detrimental for revenues to exclude bidders from the latter (somehow disadvantaged) group.

Theorem 4 *Consider environments without incumbents and with two groups of entrants ($K = 2$) and assume that the equilibria played are the ones that are most preferred by the seller. Consider a loser neutral dominant strategy auction without participation fees. If the good is under-assigned to bidders from group 1 while being over-assigned to bidders from group 2, then it can only be detrimental to exclude bidders from group 1.*

The intuition for Theorem 4 is as follows. Suppose group 1's participation rate could be decided freely while group 2's rate would adjust accordingly. Letting this participation rate vary from its equilibrium level (without set-asides) to zero, it can be shown that the total welfare -or equivalently the seller's revenue- would decrease along the path. The decrease is the result of two effects that go in the same direction: first, a marginal decrease of group 1's participation is detrimental to the welfare ceteris paribus since those bidders grab less surplus than their contribution to the social

an extra competitor. Indeed, what is used in the proof are the weaker conditions that the expected payoff of an entrant weakly (resp. strictly) decreases ceteris paribus when an extra entrant (resp. the incumbent) participates. Those sufficient conditions on the expected payoff function are actually often easier to check, e.g. in first-price auctions.

welfare. Second, there is an indirect effect of such a decrease: it increases group 2's participation which is detrimental to the welfare since those bidders grab more surplus than their contribution to the social welfare.

5.4 Applications of Theorems 2, 3 and 4

Theorems 2, 3 and 4 can be used to shed light on a variety of applications in which bidders are not treated alike. We first consider the case of procurements in which the terms of the contracts can be renegotiated after the auction stage and the incumbent would be better than the entrants at such renegotiations. We next consider the possibility that bidders may go bankrupt after the auction in which case the contract promised at the auction stage would not be honored, and we allow for asymmetries in the risk of bankruptcy. Such considerations are of primary importance in the context of procurement auctions as reported among others in Spulber (1990).

5.4.1 Asymmetric renegotiation abilities

Renegotiation is an important dimension of procurements as emphasized by Bajari, Houghton and Tadelis (2014). There is typically an important discrepancy between initial bids and final payments,³⁸ where the discrepancy comes from the fact that contracts are renegotiated due to unforeseen contingencies that require adaptation costs. Bajari et al. (2014) argue that adaptation costs are of larger magnitude than the losses due to imperfect competition at the auction stage. Importantly, firms are not on equal footing to obtain good deals at the renegotiation stage (presumably incumbents who have more familiarity are better at obtaining good deals). Such asymmetries at the renegotiation stage lead to asymmetric bidding behaviors at the auction stage despite the fact that the auction format seems to be treating all bidders in the same way.

Let us consider a very simple reduced-form model of renegotiation in the Vickrey auction. Each bidder i has some ability to renegotiate the final price, which is modeled as follows. Let $\beta_i : [r_i, \infty) \rightarrow [X_S, \infty)$ denote an increasing function such that if bidder i wins the auction and is supposed to pay $p \geq X_S$ then the effective price after renegotiation is $\beta_i^{-1}(p) \geq r_i \equiv \beta_i^{-1}(X_S)$.³⁹ Note that r_i can be interpreted as the effective reserve price bidder i faces. Due to the renegotiation stage and assuming that bidders perfectly anticipate the discrepancy between the final bid and the effective price they will pay, bidders no longer have an incentive to bid their valuation: it is now a weakly dominant strategy for bidder i to bid $\beta_i(x)$ for any valuation $x \geq r_i$ and to not participate otherwise. If all bidders have the same ability to renegotiate (i.e. the same function β) and if the corresponding effective reserve price is $r = X_S$, then the second-price auction with the reserve price X_S implements the true Vickrey auction payoffs, namely the ones corresponding to bidders' marginal contribution to the welfare, and we can still apply Theorem 1. But, when

³⁸One of the most spectacular example is Sydney opera house budgeted at an initial cost of \$7 million and ended up costing more than \$100 million (see Flyvbjerg (2005) for practical elements on cost overruns).

³⁹It does not matter that the effective price paid ex post is a deterministic function of the final auction price. We only need to interpret $\beta_i^{-1}(p)$ as the *expected* price that bidder i would pay if the final price in the auction is p .

bidders differ in terms of ability to renegotiate, there is now some inefficiency at the auction stage, and it is of interest to analyze what these inefficiencies imply in terms of the desirability of the exclusion of some bidders.

Consider two bidders i and j . We say that bidder i is a better renegotiator than bidder j if $\beta_i(x) \geq \beta_j(x)$ for any $x \geq r_j$ and if $r_i \leq r_j$. Consider first environments in which all entrants have a (common) ability to renegotiate that may differ from that of the incumbent. Specifically, let β_{ent} denote the renegotiation function for entrants that applies to all groups, and let β_{inc} denote the renegotiation function for the incumbent. If $\beta_{inc}(x) \neq \beta_{ent}(x)$ for some x in bidders' valuation distribution, then the second price auction with reserve price X_S is no longer ex post efficient. If $\beta_{inc}(x) \geq \beta_{ent}(x)$ and $r_{inc} \leq X_S = r_{ent}$, the good is over-assigned to the incumbent while the auction is efficient among entrants and the seller, and we can then apply Theorem 2.

Corollary 5.3 *If the incumbent is a better renegotiator ex-post ($\beta_{inc}(x) \geq \beta_{ent}(x)$, for any $x \geq r_{ent}$, and $r_{inc} \leq r_{ent}$) and if $r_{ent} = X_S$, then the revenue-optimal set-asides policy consists in excluding the incumbent in the second-price auction with reserve price X_S .*

Similarly, we can apply Theorem 3:

Corollary 5.4 *Assume that there is a single group of entrants who are better renegotiators ex-post than the incumbent ($\beta_{inc}(x) \leq \beta_{ent}(x)$) and $r_{ent} = r_{int} = X_S$.⁴⁰ If the seller internalizes the payoff of the incumbent, then the optimal set-asides policy for the seller consists in no-exclusion in the second-price auction with reserve price X_S .*

Consider next environments without incumbents and with two groups of entrants ($K = 2$) having asymmetric abilities to renegotiate. Let β_{ent}^i denote the function that characterizes the ability to renegotiate of entrants from group $i = 1, 2$. If $\beta_{ent}^2(x) \geq \beta_{ent}^1(x)$ (for any $x \geq r_{ent}^1$) and $r_{ent}^2 \leq X_S \leq r_{ent}^1$, then the good is under-assigned (resp. over-assigned) to the entrants from group 1 (resp. 2) and we can apply Theorem 4.

Corollary 5.5 *Suppose there are two groups of entrants, that entrants from group 2 are better renegotiators than entrants from group 1 and that $r_{ent}^2 \leq X_S \leq r_{ent}^1$. Then it is detrimental to exclude bidders from group 1 in the second-price auction with reserve price X_S .⁴¹*

Comment. In some procurements, bid subsidies⁴² are used to favor some kinds of bidders, e.g. domestic bidders or small businesses. In practice, it typically takes the form of linear bid subsidies: it means that if a favored bidder wins the auction at price p , he will pay only $(1 - \alpha)p$ where $\alpha \in (0, 1)$. A bid subsidy is then analogous to an ex-post renegotiation as just formalized and the above analysis can equally be applied to such contexts.

⁴⁰This last assumption guarantees also that excluding the entrants is not profitable since the revenue is guaranteed to be above X_S and is stuck at X_S if there is a single bidder.

⁴¹We still need an equilibrium selection. To alleviate the presentation we omit it here as in our subsequent corollaries of Theorem 4.

⁴²See Athey, Coey and Levin (2013), Krasnokutskaya and Seim (2011) and Marion (2007).

5.4.2 Asymmetric risks of bankruptcy

Aside from renegotiation, another major concern in procurement auctions is the risk of bankruptcy, and different firms may have different risks of bankruptcy (see Spagnolo (2012) and Saussier and Tirole (2015)).⁴³ Several works have formalized the risk of failure (see e.g. Zheng (2001), Board (2007), Burguet, Ganuza and Hauk (2012)) through models in which the winner has the option to not realize the project ex post at a stage where the firm gets better informed about its cost. In that literature, the risk of failure is endogenous to the contract to the extent that the exit option is exerted differently depending on the terms of the contract at the auction stage (that affect the overall profitability of the firm). By contrast we consider below a simpler model in which the risk of bankruptcy is unrelated to what happens at the auction stage. Our model fits better situations in which the risks of breakdown are not driven by the considered procurement assumed to be small in regard of the activity of the firm. For simplicity, we also assume that there are no monetary transfers if bankruptcy occurs in which case the seller is assumed to keep the good. In terms of welfare, this means that when a bidder with valuation x has a probability p to go bankrupt, then the corresponding effective valuation to be counted for the contribution to welfare, referred to as the “correct valuation”, is $(1 - p) \cdot x + p \cdot X_S$.

In an auction where payments occur only when there is no bankruptcy, the risk of bankruptcy does not play any role in the bidding incentives. In particular, in the second-price auction, for each bidder it is still a dominant strategy to bid his valuation. If all bidders have the same probability of default, then the second-price auction with reserve price X_S implements the Vickrey auction payoffs with respect to the correct valuation and we can apply Theorem 1. We now discuss cases where bidders differ in terms of risks of bankruptcy.

Consider first environments without incumbents and with two groups of entrants ($K = 2$). The probability to go bankrupt for entrants from group $i = 1, 2$ is denoted by $p_{ent}^i \in [0, 1]$. If $p_{ent}^1 \neq p_{ent}^2$ the second price auction (with $r = X_S$) is not ex post efficient. If $p_{ent}^1 < p_{ent}^2$, the good is under-assigned (resp. over-assigned) to the entrants from group 1 (resp. 2) and we can apply Theorem 4.

Corollary 5.6 *If there are two groups of entrants and if the probability of bankruptcy is lower for group 1, then it is detrimental to exclude bidders from group 1 in the second-price auction with reserve price X_S .*

Consider next environments in which all entrants have a (common) probability $p_{ent} \in [0, 1]$ to go bankrupt, and the (single) incumbent has a probability $p_{inc} \in [0, 1]$ to go bankrupt. If $p_{inc} \neq p_{ent}$, then the second price auction (with $r = X_S$) is not ex post efficient. If $p_{inc} > p_{ent}$ (resp. $p_{inc} < p_{ent}$), the good is over-assigned (resp. under-assigned) to the incumbent (according to the correct valuations) and we can apply Theorem 2 (resp. 3).

⁴³Abnormally low bids are often perceived as irregular and are discarded on the ground that the risk of failure is too high.

Corollary 5.7 *Assume that the risk of breakdown is larger (resp. smaller) for the incumbent. If the seller's maximizes revenue (resp. internalizes the payoff of the incumbent), then the optimal set-asides policy for the seller consists in excluding the incumbent (resp. in no-exclusion) in the second-price auction with reserve price X_S .*

Comments: 1) In a number of instances, the plausible assumption is that the incumbent has a smaller risk of breakdown than entrants so that Corollary 5.7 is a call for no-exclusion if the seller internalizes a large enough share of the incumbent's payoff and a call for exclusion only to the extent that the bankruptcy risks of the incumbent and the entrants are not too dissimilar and that the seller does not internalize much the rents of the incumbent. 2) In our environment with risks of breakdown, the correct Vickrey auction would assign the good to the bidder with the highest correct valuation. More precisely, the bidder with the highest correct valuation $(1-p) \cdot x + p \cdot X_S$ should win the good (provided that $x \geq X_S$) and pay⁴⁴ his contribution to the welfare $x' + \frac{(p-p')}{(1-p)}(x' - X_S)$ where x' and p' correspond respectively to the valuation and probability of bankruptcy of the bidder with the second highest correct valuation (provided $x' \geq X_S$). For such an efficient auction, we could apply Theorem 1 directly. More generally, if the auction does not take into account appropriately the relative risk of bankruptcy beyond the point where it over-assigns the good to the incumbent (resp. under-assigns the good to the entrants from group 1), then Theorem 2 (resp. 4) extends. 3) Our environment with risk failure is actually equivalent to the auction models used for advertisement slots on Internet (as used by Google and most publishers) under a pay-per-click system (Agarwal, Athey and Yang, 2009). Given that there is a huge heterogeneity in terms of the probability of clicks (or conversion rates), winners are no longer ranked according to their bid per click as it used to be but rather according to their bid times the estimated probability that they receive a click. The evolution of the mechanism can be roughly interpreted as a move from the (inefficient) Vickrey auction to the Vickrey auction with respect to the correct valuation.

5.5 General mechanisms in Myersonian environments

So far we have considered auction formats in which bidders have a weakly dominant strategy. The analysis extends straightforwardly to general auction formats when valuations are drawn independently across bidders (Myersonian environments) and the equilibrium allocation is the same as the one arising in the previously considered mechanisms. In particular, Theorem 1 extends straightforwardly to any mechanism which is payoff-equivalent to the Vickrey auction from an ex ante perspective and for any possible entry profile μ and any vector of incumbents I so that the expected profit of the incumbent i (resp. a group k entrant) is still equal to $\Pi_i^{inc}(\mu, I)$ (resp. $\Pi_k^{ent}(\mu, I)$). On the one hand, the revenue is the same as in the Vickrey auction for any

⁴⁴Indeed, the payment occurs only when the winner is not bankrupt so that the expected payment is the previous figure multiplied by $(1-p)$.

entry profile. On the other hand, since the equilibrium free entry conditions are determined by the payoffs of the entrants whose expressions remain unchanged, the set of equilibrium entry profiles is the same as in the Vickrey auction. From the well-known “payoff equivalence Theorem” (see e.g. Milgrom, 2004), in a Myersonian environment, any mechanism which assigns the good efficiently and leaves no rents to buyers with null valuation is payoff-equivalent to the Vickrey auction. An example of such a mechanism is the first-price auction with the reserve price X_S in a Myersonian setup with symmetric buyers but also under the extra assumption that the set of entrants is publicly observed before the bidding stage. In particular, with a single incumbent, we still get that it would be profitable to exclude him.⁴⁵

More generally, in a Myersonian environment, we know that the rents of the various agents are fully determined (up to some constants) by the assignment rule. If a given auction induces (in equilibrium) an assignment rule that can be implemented in dominant strategy,⁴⁶ and if we can apply Theorems 2 or 4 to the payoff equivalent dominant strategy auctions, then the results apply to our given auction. For example, in environments with a single incumbent and if the auction induces a $\phi(b, r)$ -assignment rule with $b(x) \geq x$ and $r \leq X_S$, then the optimal set aside policy consists in excluding the incumbent and keeping the entrants.⁴⁷ We apply this principle to the study of first-price auctions.

First-price auctions with or without the right-of-first-refusal

A form of explicit discrimination sometimes encountered in procurement auctions is the right-of-first-refusal: it consists in letting a special (or preferred) bidder match the final highest bid as it is analyzed in Burguet and Perry (2006) in a procurement setup.⁴⁸ We consider below the effect of excluding the incumbent assumed to enjoy the right-of-first-refusal. At first, it might seem odd to simultaneously assume that the incumbent can be excluded and can enjoy a right-of-first-refusal.

⁴⁵If the set of entrants is not observed, then it creates an asymmetry between the incumbent and the entrants: an entrant expect to face one more competing bidder than the incumbent and thus bid more aggressively than the incumbent which induces inefficiencies. Formally, if $G_\mu(\cdot)$ (resp. $G_\mu^I(\cdot)$) denote the CDF of the highest bid among the entrants (resp. the bid of the incumbent), then in the Poisson model the distribution of the highest competing bid of any entrant is $b \rightarrow G_\mu(b) \cdot G_\mu^I(b)$, which first-order stochastically dominates $b \rightarrow G_\mu(b)$ the distribution of the highest competing bid of the incumbent. If bidders do not receive extra information additional to their valuation, then entrants should bid more aggressively than the incumbent in the auction and the incumbent is thus disadvantaged which is a countervailing force against his exclusion.

⁴⁶From Mookherjee and Reichelstein (1992), in our simple single-good auction setup: dominant strategy implementation is feasible if and only if the probability that a bidder wins the good is non-decreasing in his valuation for any set of valuation of his opponents. E.g. $\phi(b, r)$ assignment rules are implementable in dominant strategy.

⁴⁷The class of $\phi(b, r)$ -allocation rules excludes allocation rules that depend on the realization of the set of entrants (e.g. the number of entrants). Nevertheless, the argument would also extend if a different function b is used (to characterize the assignment rule) depending on the set of entrants (e.g. the number of entrants).

⁴⁸In a procurement, it corresponds to a right to match the lowest bid. This right is often observed in procurements either through an explicit right or an implicit one (see Lee (2008) for examples of industries where it is a common practice). Note also that corruption in procurements has also been modeled as a bribery auction where the winner obtains a right of first refusal (Compte et al., 2005). In procurements for public transportation contracts in London, after reviewing the bids, the regulator can ask the incumbent for a second offer (if his offer is close to the winning bid) for him to win the bid (Amaral et al. (2009)). In the English auctions for cricket players in the Indian Premier League, the team that owns the auctioned player in the previous season has an equivalent “right to match” the winning bid (Lamy et al. (2016)). Those two examples illustrate that the right-of-first-refusal is typically attributed to a bidder than can be viewed as an incumbent according to our framework.

Our preferred interpretation is that the right-of-first-refusal captures a form of implicit corruption between the representative of the authority and the incumbent, and the question for the authority (not its representative) is whether it may be beneficial to not let the incumbent participate (taking as given the reduced form effect of corruption if the incumbent can participate).

In a first price auction with reserve price X_S , assuming that entrants simultaneously choose to enter and submit a bid,⁴⁹ if the incumbent has a right-of-first-refusal, then the equilibrium assignment rule is distorted from the efficient assignment by assigning the good too often to the incumbent. More precisely, if we let $\beta : [X_S, \infty) \rightarrow [X_S, \infty)$ denote the equilibrium bid function of the entrants (where $\beta(x) < x$ for $x > X_S$ as a result of bid shading), then the incumbent will exert his right-of-first-refusal and win the good whenever the valuation of the incumbent is in the interval $(\beta(x), x)$ where x is the highest valuation among the entrants, and this is inefficient. Thus, the equilibrium allocation is such that the good is over-assigned to the incumbent, and, in a Myersonian environment, we obtain as corollary of Theorem 2 that the insight found in Example 1b applies more generally:

Corollary 5.8 *In a Myersonian environment, if the incumbent has a right-of-first-refusal in the first-price auction with reserve price X_S , then the revenue-optimal set-asides policy consists in excluding the incumbent.*

Remark. The right-of-first-refusal can also be used in second-price/English auctions as analyzed in Bikhchandani, Lippman and Reade (2005). The equilibrium analysis is then straightforward: it is a dominant strategy for the bidders who do not have this right to bid up to their valuation, while the bidder with this right matches the final price if it is below his valuation. If the reserve price is X_S , the auction obviously over-assigns the good to the incumbent while the mechanism is a dominant-strategy auctions without participation fees so that we can apply Theorem 2 directly.

To conclude this section, we discuss informally what happens to our exclusion insights in standard first-price auctions (with no right of first refusal) and still with the reserve price set at X_S , assuming that potential entrants are symmetric but that there may be asymmetries between the incumbent and the entrants. In a (standard) first-price auction where the number of entrants is disclosed, whether the good is over- or under-assigned to the incumbent depends on the relative strength of the incumbent and of the entrants' valuation distributions. From Maskin and Riley (2000) and Lebrun (1999), we know that a bidder who has a “weaker” (resp. “stronger”) ⁵⁰ valuation distribution bids more aggressively, and thus accordingly the good is over-assigned (resp. under-assigned) to the weak (resp. strong) bidder following our terminology. As a result,

⁴⁹This guarantees that all entrants from the different groups use the same bidding function as detailed in Jehiel and Lamy (2015).

⁵⁰Maskin and Riley (2000) and Lebrun (1999) need a stronger notion of dominance than first-order stochastic dominance: it is reverse hazard rate dominance. The incumbent is said to be weaker (resp. stronger) than the entrants if $\frac{f^I(x)}{F^I(x)} \geq$ (resp. \leq) $\frac{f(x)}{F(x)}$ for any $x \in (x, \bar{x}]$.

we obtain that it is always revenue-optimal to exclude the incumbent if he is weak while no-exclusion is welfare-optimal if he is strong.⁵¹ At the other extreme, note that a very strong incumbent would completely discourage entry, which would then make exclusion profitable. For other cases in which the incumbent is stronger than entrants but to a more moderate extent, our analysis does not allow to conclude whether excluding the incumbent would be good for revenues.

Concerning the exclusion of entrants (when there are only two groups of entrants so that we can apply Maskin and Riley (2000) and Lebrun (1999)) and assuming there are no incumbents and the set of entrants is disclosed -not only the number of entrants but also their group identities-, we obtain then that it is always detrimental to exclude the entrants from the stronger group. Turning to the policy debate on set-asides in public procurements and given that SMEs are typically viewed as having weaker valuation distributions, such an observation pleads thus against set-asides for SMEs when large firms can be considered as entrants in our terminology (i.e., when they do not participate for sure).

6 The Vickrey auction with multiple incumbents

In order to understand better the multiple incumbents case, we come back to the Vickrey auction, and we consider the case of symmetric potential entrants. Given the restriction to a single group of entrants, we alleviate notation, and simplify the equilibrium entry rate $\mu^*(I, E)$ into $\mu^*(I)$. With multiple incumbents, if we plug (17) for $i \in I$ into the expression of the revenue (14), we get

$$R(\mu^*(I), I) = TW(\mu^*(I), I_{-i}) - \sum_{j \in I_{-i}} \Pi_j^{inc}(\mu^*(I), I) \quad (21)$$

and the revenue effect of excluding incumbent i can be written as:

$$R(\mu^*(I_{-i}), I_{-i}) - R(\mu^*(I), I) = \overbrace{TW(\mu^*(I_{-i}), I_{-i}) - TW(\mu^*(I), I_{-i})}^{\geq 0} - \sum_{j \in I_{-i}} \underbrace{[\Pi_j^{inc}(\mu^*(I_{-i}), I_{-i}) - \Pi_j^{inc}(\mu^*(I), I)]}_{\geq \text{ or } \leq 0 ?}. \quad (22)$$

There are two conflicting effects of excluding incumbent i . On the one hand, it increases the first term in (21) since $TW(\mu^*(I), I_{-i}) \leq TW(\mu^*(I_{-i}), I_{-i})$. In words, it increases the total welfare minus the rents of incumbent i . This is the analog of the (positive) effect of exclusion with a single incumbent that we have previously highlighted. On the other hand, there is a novel effect at work. Excluding incumbent i can also have an impact on the rents of the remaining incumbents. The sign of this second effect is ambiguous without additional restrictions. *Ceteris paribus* (without taking into account the effect on the entry rate), the presence of incumbent i has

⁵¹Instead of applying Theorem 3, it is sufficient to check that the payoff of an entrant does not increase when there are more entrants (see footnote 37) which holds in first-price auctions (see Arozamena and Cantillon (2004) for details on the monotonicity when competition increases).

a negative impact on the rents of the other incumbents. However, excluding incumbent i has also an impact on the entry rate. Intuitively, without incumbent i , the participation rate of entrants should be larger (this is true if potential entrants are symmetric, otherwise the entrants of some group(s) may possibly participate less), which should attenuate if not counterbalance the previous negative impact on the seller's revenue. The rest of this Section elaborates on this intuition.

6.1 A special class of incumbents

In a simple class of incumbents' valuation distributions, we are able to show that the revenue advantage of excluding the incumbents extends to the multi-incumbent case. Specifically, consider the following assumption on bidders' valuation distributions.

Assumption A 1 *Potential entrants are symmetric with valuations distributed according to $F(\cdot|\cdot)$ and for each incumbent $i \in \mathcal{I}$, there exists $\lambda_i \geq 0$ such that $F_i^I(x|z) = e^{-\lambda_i \cdot (1-F(x|z))}$ for each x, z .*

An interpretation of this class of distributions is that the valuation of incumbent i can be viewed as being the highest valuation among potential entrants entering according to the Poisson distribution with λ_i entrants on average. This implies that if the incumbent i is substituted by an average of λ_i extra entrants, then the distribution of the highest competing bid remains the same from the viewpoint of any new entrant contemplating whether or not to enter. Formally, under A1 we have

$$\Pi^{ent}(\mu + \lambda_i, I_{-i}) = \Pi^{ent}(\mu, I). \quad (23)$$

This further implies that $\mu^*(I_{-i}) = \mu^*(I) + \lambda_i$ if $\mu^*(I) > 0$. Combined with the analog of the equality (23) but for the incumbents $j \neq i$, we obtain that $\Pi_j^{inc}(\mu^*(I_{-i}), I_{-i}) = \Pi_j^{inc}(\mu^*(I), I)$ for any $j \neq i$ so that it is profitable to exclude bidder i . Given that $\mu^*(\mathcal{I}) \leq \mu^*(I)$ for any $I \subseteq \mathcal{I}$, if $\mu^*(\mathcal{I}) > 0$, our argument can be repeated for any set of incumbents, thereby allowing us to conclude:

Proposition 6.1 *Under Assumption 1, if $\mu^*(\mathcal{I}) > 0$, then the revenue-optimal set-asides policy in the Vickrey auction consists in excluding all incumbents.*

Remark. Note that the welfare loss from full exclusion is equal to $(\mu^*(\mathcal{I}) - \mu^*(\emptyset)) \cdot C$. However, this loss is more than compensated by the rents of the incumbents. For incumbent i , his equilibrium rent is always larger than $\lambda_i \cdot C$ (provided that the entry rate of potential entrants is strictly positive). The interpretation here is that an incumbent can be viewed as a ring of an average of λ_i buyers (and where the size of the ring follows a Poisson distribution). The ring sends to the auction only the buyer with the highest valuation. By contrast, if the various buyers in the ring would have behaved competitively then those buyers would be exactly in the same situation as the potential entrants and their expected gross payoff in the auction would be C in equilibrium

(it is thus the cooperative behavior through the ring that explains why the incumbent is making extra rents).

6.2 Asymmetric bidders

To cover a larger range of valuation distributions for incumbents, we consider the following set of assumptions that will be used to analyze the effect of excluding incumbent $i \in I$:

Assumption A 2 1. We are in a Myersonian environment with symmetric entrants ($K = 1$) whose valuations are distributed according to $F(\cdot)$,

2. The function $x \rightarrow \frac{1}{F_j^I(x)} \cdot \frac{1-F_j^I(x)}{1-F(x)}$ is (strictly) decreasing on (\underline{x}, \bar{x}) for any $j \in I_{-i}$,

3. The function $x \rightarrow -\frac{\log[F_i^I(x)]}{1-F_i^I(x)} \cdot \frac{1-F_i^I(x)}{1-F(x)}$ is (strictly) decreasing on (\underline{x}, \bar{x}) .

Remarks. Assumption 2 holds in the special case in which the valuation distribution of the incumbents is also F . A general simple sufficient condition guaranteeing that the monotonicity conditions hold in Assumption 2 is given in the Appendix: It is satisfied if the distribution of the incumbents can be interpreted as (possibly different) mixtures of ring of entrants of various sizes (i.e. such that the valuation is distributed according to F^k if the ring is of size $k \geq 1$). Observe that when Assumption 1 holds, Assumption 2 is violated just at the margin insofar as the monotonicity in the third requirement fails to be strict.⁵²

Proposition 6.2 Under Assumption 2, in the Vickrey auction, the rents of the non-excluded incumbents I_{-i} increase when incumbent i is excluded.

Hence, under the conditions of Proposition 6.2, there is a non-trivial trade-off on the effect of revenues of excluding incumbent i . On the one hand, excluding incumbent i enhances the welfare net of incumbent i 's rents as in our single incumbent case. On the other hand, it increases the rents of the non-excluded incumbents. While in general it is not clear in which direction the trade-off might go, we note now that when incumbent i is sufficiently small/weak (everything else being equal), it is not good to exclude i .

To see this, observe that $\mu^*(I_{-i}) - \mu^*(I)$ must be small if incumbent i is sufficiently weak (thereby winning very rarely the auction and thus affecting only marginally the entry decisions). Developing (22) reveals that the first (beneficial) effect is approximately equal to

$$\frac{\partial TW(\mu^*(I_{-i}), I_{-i})}{\partial \mu} \cdot [\mu^*(I_{-i}) - \mu^*(I)] + \frac{\partial^2 TW(\mu^*(I_{-i}), I_{-i})}{(\partial \mu)^2} \cdot \frac{[\mu^*(I_{-i}) - \mu^*(I)]^2}{2} = \frac{\partial^2 TW(\mu^*(I_{-i}), I_{-i})}{(\partial \mu)^2} \cdot \frac{[\mu^*(I_{-i}) - \mu^*(I)]^2}{2}$$

⁵²More precisely it is the third requirement that is violated under Assumption 1 (assuming we are in a Myersonian environment) given that $-\frac{\log[F_i^I(x)]}{1-F_i^I(x)} \cdot \frac{1-F_i^I(x)}{1-F(x)} = \lambda_i$ which does not depend on x . By contrast, one can check that the second requirement in A2 holds since $x \rightarrow \frac{e^{-\lambda(1-F(x))}-1}{1-F(x)}$ is (strictly) decreasing (by using the inequality $e^{1+x} \geq x$).

since $\mu^*(I_{-i}) \in \text{Arg max}_{\mu \in \mathbb{R}_+} TW(\mu, I_{-i})$. Hence, the first channel is of second order due to the welfare-maximizing entry rate when i is absent. By contrast, the second (detrimental) effect on other incumbents' rents is of first order,⁵³ thereby allowing us to conclude:

Proposition 6.3 *Assume incumbent i 's valuation distribution takes the form $F_i^I(x) = [G(x)]^\lambda$ for some $\lambda > 0$ and where $G(\cdot)$ is a CDF. Let Assumption 2 hold.⁵⁴ If λ is small enough then the revenue in the Vickrey auction reduces when incumbent i is excluded.*

This proposition extends the insight developed in Example 1d illustrating that small/weak incumbents should not be excluded.

7 Further insights

7.1 Beyond private values: the case of an informed incumbent

In interdependent value contexts, rational bidders tend to bid more cautiously when the number of participants increases so as to internalize the winner's curse. Bulow and Klemperer (2002) suggest that it can then be beneficial for the seller to limit the total number of potential bidders when bidders are symmetric or to exclude a strong bidder whose presence would reduce the competition by exacerbating others' response to the winner's curse.⁵⁵ As we now illustrate, our exclusion principle is typically reinforced if the information held by the incumbent affects the valuation of other participants due to an informational advantage of the incumbent. Roughly, the intuition as to why it is still good to exclude the incumbent in this case is as follows: Entrants tend to bid more cautiously as compared with the situation in which they would know the signal of the incumbent so as to internalize the winner's curse. This in turn ensures that the incumbent gets a rent that is larger than his marginal contribution to the welfare. The conclusion follows then by using arguments similar to those developed above in the private value case.

To formalize this, consider the same model as in Section 2 but with the difference that what we call valuations are now private signals. For the incumbent, we still assume that his signal, denoted by x_I , coincides with his valuation. However, for a given entrant, we assume that if he receives the signal x_E then his valuation is now $H(x_E, x_I)$ where $H(\cdot, \cdot)$ is increasing in both arguments, strictly increasing in its first argument with the derivative with respect to its second argument being less than one. Then without loss of generality, we can normalize the entrants' signals so that $H(x, x) = x$.⁵⁶

To make the analysis simpler, we consider the ascending (button) auction with no reserve price in which the price raises continuously, bidders may quit at any time, and the auction stops

⁵³For this, it is crucial that the monotonicity in the third requirement is strict.

⁵⁴Note that if A2 holds (with respect to incumbent i) for a given $\lambda > 0$ then it will be so for each $\lambda > 0$.

⁵⁵Compte and Jehiel (2002) show that even the welfare may increase when some bidders are excluded.

⁵⁶As in Milgrom and Weber (1982), we consider a model involving both private and common values components. However we depart from their symmetry assumption and as in Engelbrecht et al. (1983) we consider that only one bidder is informed about the common value component. Such models have been developed for oil and gas leases auctions (Hendricks and Porter, 1988) and more recently for Internet ad slots auctions (Abraham et al., 2013).

when there is one bidder left who has then to pay the current price. We also assume that bidders observe the identity of the remaining active bidders, and that the seller's reservation value is null ($X_S = 0$). The (weakly dominant) equilibrium strategy of the incumbent consists in remaining active up to his valuation x_I . For the entrants, the decision to exit or to remain active at price p depends on whether the incumbent is still active or not. If the incumbent has dropped from the auction thereby revealing his valuation x_I through his exit time, an entrant with signal x remains active up to $H(x, x_I)$. If the incumbent is still active, which only reveals that $x_I \geq p$, an entrant with signal x remains active up to x .⁵⁷ Such strategies constitute an ex-post equilibrium, and the good is assigned efficiently. Without the incumbent, we are back to the Vickrey auction in a private value setting. However, it is no longer the case that the payoff of the incumbent corresponds to his contribution to welfare, i.e. (17) no longer holds in the interdependent value case. The rent (resp. the marginal contribution to the welfare) of the incumbent with valuation x_I when the largest signal among entrants is x_E can be written as $x_I - x_E$ (resp. $x_I - H(x_E, x_I)$) if $x_I > x_E$ and 0 otherwise (and obviously both the rent and the contribution to welfare coincide when there are no entrants). Since $H(x_E, x_I) \geq H(x_E, x_E) = x_E$ if $x_E \leq x_I$, the expected rent of the incumbent outweighs his expected contribution to welfare, which in turn allows us to extend Theorem 1 to this interdependent value setting:

Proposition 7.1 *In an English auction with no reserve price, when the seller's reservation value is null and when valuations may depend on the incumbent's information as described above, the revenue-optimal set-asides policy consists in excluding the incumbent and allowing all groups of entrants to participate.*

7.2 Set-asides versus split-awards

As advocated by Milgrom (2004), split-awards constitute alternative tools to promote entry in situations with asymmetric competitors.⁵⁸ Instead of selling the good as a single lot, the seller can split the good into several lots (possibly of different sizes) typically requiring that any given bidder can buy at most one lot. While split-awards may have good properties in terms of reducing the rents left to the incumbent(s), we note in the following result that when there is a single incumbent, excluding the incumbent is always preferable to using even cleverly designed split-awards.

Formally, a split-award is characterized by $\alpha = (\alpha_1, \dots, \alpha_K) \in (0, 1]^K$ with $\sum_{k=1}^K \alpha_k = 1$ specifying that the good is split into K lots of sizes $\alpha_1, \dots, \alpha_K$ with the requirement that a given bidder cannot be assigned more than one lot. Buyers' valuations are assumed to be linear in the

⁵⁷This follows from usual marginal considerations as developed in Milgrom and Weber (1982). This is also where the assumption $\frac{\partial H(x_E, x_I)}{\partial x_I} < 1$ is used.

⁵⁸The split-award literature (Anton and Yao (1989) and Gong, Li and McAfee (2011)) has emphasized the possible benefits in terms of pre-participation investments rather than entry. In a very different context but in a related vein, Gilbert and Klemperer (2000) show that quantity rationing can be profitable for a monopolist so as to increase low-value consumer's incentives for pre-purchasing investments.

size of the lot. That is, a buyer with valuation x attaches a valuation $\alpha_k \cdot x$ to the lot k with size α_k . For any split-award α , we can define an associated generalized Vickrey auction (Edelman, Ostrovsky and Schwarz, 2007) in which buyers get their marginal contribution to the welfare given the allotment constraints imposed by the split-award α (for example, if the split-award involves symmetric lots, i.e. $\alpha_1 = \dots = \alpha_K = \frac{1}{K}$, then the associated generalized Vickrey auction is the $K + 1^{th}$ -price auction).

Jehiel and Lamy (2015) show that the property that equilibrium entry rates must maximize the total welfare (here Lemma 3.1) extends to split-award generalized Vickrey auctions. As argued in Section 4.3, we obtain then that Theorem 1 remains valid in such environments, namely that the revenue can only increase if the incumbent is excluded.⁵⁹ But, once the incumbent is excluded, Jehiel and Lamy (2015) imply that the standard Vickrey auction (which is optimal among all possible mechanisms) outperforms any split-award auction. We conclude that

Proposition 7.2 *When there is a single incumbent, the revenue in any split-award generalized Vickrey auction with the incumbent is dominated by the Vickrey auction in which the incumbent is excluded.*

Remark. We note that one cannot apply our analysis with multiple incumbents to split-award generalized Vickrey auctions: in particular, facing an incumbent with the distribution $F_i^I(x|z) = e^{-\lambda_i \cdot (1-F(x|z))}$ is no longer equivalent to facing an average of λ_i entrants with the distribution $F(x|z)$ (following a Poisson distribution) due to the constraint that a bidder can get at most one lot.

7.3 Set-asides versus fees/subsidies

If the seller were free to charge any fee including incumbents, the seller's revenue would be aligned with the total welfare assuming incumbents have no private information at the time the fee is charged, and thus set-asides would not be optimal.⁶⁰

In this section, we develop a more restrictive view on fees: we consider that the seller can only tax or subsidize entry. In some procurements, there are some funds dedicated to reimburse partially the physical participation costs of the bidders (in an attempt to reduce the barriers to entry).⁶¹ For example, in the merger case considered in footnote 1, the French antitrust authority decided to create a fund (to be financed by the merged entity) aimed at boosting competition by reimbursing the participation costs of those firms that were not incumbents. Below, we have

⁵⁹More generally, we can apply our exclusion insight to any auction format that can be reinterpreted as a pivot/generalized Vickrey auction of a different problem (assuming the seller can coordinate entry on her most preferred equilibrium).

⁶⁰With such fees, it is as if the seller internalizes the rents of the incumbents. However, such a solution, which outperforms Jehiel and Lamy's (2015) optimal auction, does not seem realistic because it will stand in conflict with participation constraints.

⁶¹See Gal and al. (2007) for such a model that relies crucially on the fact that those participation costs are verifiable to some extent, namely that we cannot have opportunistic bidders who enter the auction only to collect those subsidies (which in our Poisson model would raise existence of equilibrium issues).

implicitly in mind that subsidizing entry is easier to implement than imposing an entry tax (which may in some circumstances lead to the violation of participation constraints).

Similarly to set-asides, subsidies for entrants are an instrument that allows to get closer to the optimal revenue characterized in Jehiel and Lamy (2015) insofar as it allows the seller to reduce the incumbents' rents.

We note from (21) that the following expression

$$\max_{\mu \in \mathbb{R}_+^K} R(\mu, I) = \max_{\mu \in \mathbb{R}_+^K} \left\{ TW(\mu, I_{-i}) - \sum_{j \in I_{-i}} \Pi_j^{inc}(\mu, I) \right\} \quad (24)$$

is an upper bound on the revenue that can be reached with the set of incumbents I (assumed to include incumbent i) when the seller can charge fees/subsidies to entrants (and only them) on the top of the Vickrey auction. As it turns out, if the seller is free to charge any group-specific fee e_k to entrants from group k ($e_k < 0$ corresponds to a subsidy), such a bound can be reached (this amounts to adjusting the fees/subsidies so that the required μ are obtained, formally the entry fee charged to a group k entrant should be set at $e_k = \Pi_k^{ent}(\mu, I)$ for each $k \in \mathcal{E}$).⁶²

If incumbent i is excluded, the revenue is bounded from above by

$$\max_{\mu \in \mathbb{R}_+^K} R(\mu, I_{-i}) = \max_{\mu \in \mathbb{R}_+^K} \left\{ TW(\mu, I_{-i}) - \sum_{j \in I_{-i}} \Pi_j^{inc}(\mu, I_{-i}) \right\} \quad (25)$$

and this bound is reached with no fees required if there is a single incumbent (i.e. $I_{-i} = \emptyset$). Since $\Pi_j^{inc}(\mu, I) \leq \Pi_j^{inc}(\mu, I_{-i})$ for any $j \in I_{-i}$, we obtain the following result:

Proposition 7.3 *Consider an environment in which the seller is free to post any entry fees to the entrants in the Vickrey auction. 1) If there is a single incumbent, then fees do not outperform set-asides: the revenue-optimal set-asides and fee policy consists in excluding the incumbent, keeping all kinds of entrants and having no fees. 2) Excluding some incumbents is always (weakly) dominated by the policy that consists in imposing optimal fees and keeping the incumbent. 3) When entrants are homogeneous, the optimal fee takes the form of a partial reimbursement.*

Comments. 1) To the extent that the cost C_k can be interpreted as the outside option value of a group k bidder, it may differ from the physical participation cost. The required subsidy in result 3 of Proposition 7.3 while smaller than C_k need not be smaller than the physical participation cost. Hence, set-asides may in some cases strictly dominate subsidies whenever these are required to be smaller than the physical participation costs. 2) When there is a single incumbent, we note that there are two ways of reaching the optimal revenue: either by excluding the incumbent or alternatively by keeping the incumbent but imposing a fee that implements the optimal entry rate. However, the latter policy requires a detailed knowledge of how valuations are distributed in contrast to the exclusion policy.

⁶²We stress that this bound is lower than the optimal revenue in Jehiel and Lamy (2015) whose analysis allows implicitly any kinds of fees for entrants but also richer forms of discrimination.

8 Conclusion

When the designer is free to discriminate as she wishes, set-asides are not optimal as shown by Myerson (1981) when the set of participants is exogenous or by Jehiel and Lamy (2015) when entry is endogenous for some groups of bidders. Instead, in the optimal auction, some bidders (the strong bidders in Myerson (1981) or the incumbents in Jehiel and Lamy (2015)) must be handicapped relative to others but not to the point that they should be excluded. It should be mentioned that the exact derivation of the optimal auction typically requires fine knowledge of the distributions of valuations (including that the valuations of those bidders who should be handicapped are drawn independently of other bidders' information), which stands in contrast to our exclusion principle result obtained in the one incumbent case that holds irrespective of any distributional assumptions and is thus immune to the Wilson critique (see Wilson, 1987). Beyond the robustness of our exclusion principle obtained with a single incumbent (and that extends partially to the case of multiple incumbents), we believe that in a number of practical cases, possible or legal discrimination cannot take the exact form required by the optimal auction, and it is then of interest to study when simple discriminatory tools such as set-asides can be revenue-enhancing as studied in this paper.

Our analysis has abstracted away from dynamic considerations. Two different lines of research could be pursued in relation to this. First, keeping the same underlying economic setup, we could allow the seller to employ dynamic mechanisms so as to better coordinate the entry decisions of bidders. While such mechanisms (if feasible - they may in some cases be considered to rely on illegal discrimination) could improve efficiency (as in Bergemann and Välimäki's (2010) dynamic pivot mechanism), it is not so clear what the effect on revenues would be as illustrated by the different insights obtained by Bulow and Klemperer (2009) and Roberts and Sweeting (2013). The welfare benefits from coordination could be counterbalanced by the fact that early entrants would now enjoy some rents giving them a status similar to that of the incumbents in our (static) framework. Second, we could embed the present static economic environment into a dynamic one in which bidders would make some pre-participation investments (e.g. information acquisition as in Bergemann and Välimäki (2002) or upgrade in the type distribution as in Arozamena and Cantillon (2004)) anticipating the future rents associated to those. The study of these extensions is left for future research.

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Appendix

Example 1d (continued)

Assume x_I and x_E are deterministic variables with $x_E > x_I + C$. We also make the normalization $C = 1$. Let $R_{with-1weak-Inc}$ denote the expected revenue of the seller if the strong incumbent (i.e. the one which has valuation x_I for sure) is excluded. Let $\alpha = \frac{x_E}{x_E - x_I} > 1$ and $\beta = \alpha + \epsilon - \alpha \cdot \epsilon \in [1, \alpha]$. Straightforward calculations lead to

$$R_{without-Inc} = x_E - (1 + \ln[x_E]),$$

$$R_{with-1weak-Inc} = x_E - \left(\frac{\alpha}{(1 - \epsilon)\alpha + \epsilon} + \ln[(1 - \epsilon)x_E + \epsilon(x_E - x_I)] \right)$$

and

$$R_{with-2-Inc} = x_E - \left((1 - \epsilon)\alpha + \epsilon + \ln[x_E - x_I] \right).$$

We have then:

$$R_{with-2-Inc} - R_{with-1weak-Inc} = \frac{\alpha}{\beta} - \beta + \ln[\beta]$$

For a given $\alpha > 1$, the difference $R_{with-2-Inc} - R_{with-1weak-Inc}$ is decreasing in β or equivalently increasing in ϵ . On the whole, we obtain that starting from the two incumbents situation, it is better to exclude (resp. to keep) the strong incumbent if ϵ is small (resp. large) enough.

We have then:

$$R_{with-2-Inc} - R_{without-Inc} = 1 - \beta + \ln[\alpha]$$

For a given α , the difference $R_{with-2-Inc} - R_{without-Inc}$ is decreasing in β or equivalently increasing in ϵ . On the whole, we obtain that starting from the two incumbents situation, it is better to exclude (resp. to keep) both incumbents if ϵ is small (resp. large) enough.

This example allows us to cook situations where $R_{with-2-Inc} < R_{without-Inc}$ and $R_{with-1weak-Inc} > R_{with-2-Inc}$, i.e. such that starting from two incumbents it is detrimental to exclude each incumbents separately but would be profitable to exclude them jointly. We can check that this works if $\alpha = 20$ and $\beta = 4$. \diamond

Proof of Lemma 3.1

Below, we show more generally that $J(I, E) = \text{Arg max}_{\mu \in \mathbb{R}_+^K | \mu_k = 0 \text{ if } k \notin E} TW(\mu, I) \neq \emptyset$ for any set-asides policy (I, E) . As a by-product, we will also obtain the existence of our equilibrium notion.

Since the support of valuation distributions are bounded by \bar{x} , we have then $W(N, I) \leq \bar{x}$. If $\mu_k > \frac{\bar{x}}{C_k}$, then we have $TW(\mu, I) < 0 \leq TW((0, \dots, 0), I)$. When maximizing the continuous

function $\mu \rightarrow TW(\mu, I)$ over $\{\mu \in \mathbb{R}_+^K | \mu_k = 0 \text{ if } k \notin E\}$, there is thus no loss of generality to limit ourselves to the compact subset $\{\mu \in [0, \frac{\bar{x}}{C_k}]^K | \mu_k = 0 \text{ if } k \notin E\}$. As a continuous function on a compact set reaches a maximum, we obtain then that $\text{Arg max}_{\mu \in \mathbb{R}_+^K | \mu_k = 0 \text{ if } k \notin E} TW(\mu, I) \neq \emptyset$.

Using standard calculations from auction theory, conditional on the realization z of Z , a buyer with valuation $u \geq X_S$ who participates in the auction against the profile $N \in \mathbb{N}^K$ of entrants and the set of incumbents I will receive the expected payoff of $\int_{X_S}^u \prod_{k=1}^K [F_k(x|z)]^{n_k} \cdot \prod_{i \in I} F_i^I(x|z) dx$.⁶³ The corresponding (interim) payoff of a group k buyer from entering such an auction, i.e. before knowing what his valuation will be and the realization of z , is given (after simple calculations) by

$$V_k^{ent}(N_{+k}, I) = \int_{X_S}^{\bar{x}} (F^{(1:N \cup I)}(x) - F^{(1:N_{+k} \cup I)}(x)) dx. \quad (26)$$

After an integration per part, the expected (interim) welfare can be expressed as

$$W(N, I) = X_S + \int_{X_S}^{\bar{x}} (1 - F^{(1:N \cup I)}(x)) dx. \quad (27)$$

Combining (26) and (27), we obtain

$$W(N_{+k}, I) - W(N, I) = V_k^{ent}(N_{+k}, I). \quad (28)$$

In words, we have the fundamental property of the Vickrey auction applied to a potential entrant: his payoff corresponds to his marginal contribution to the welfare. We note that $\frac{\partial P(N|\mu)}{\partial \mu_k} = -P(N|\mu)$ if $n_k = 0$ and $\frac{\partial P(N|\mu)}{\partial \mu_k} = -P(N|\mu) + P(N_{-k}|\mu)$ if $n_k \geq 1$. For any $k \in \mathcal{E}$, we have then $\frac{\partial TW(\mu, I)}{\partial \mu_k} = \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot [W(N_{+k}, I) - W(N, I)] - C_k$. From (15), we obtain from an ex-ante perspective that

$$\frac{\partial TW(\mu, I)}{\partial \mu_k} = \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot V_k^{ent}(N_{+k}, I) - C_k = \Pi_k^{ent}(\mu, I). \quad (29)$$

As a corollary, we obtain then that having the first-order conditions (of local optimality) $\frac{\partial TW(\mu, I)}{\partial \mu_k} \underset{\text{(resp. } \leq \text{)}}{=} 0$ if $\mu_k \underset{\text{(resp. } = \text{)}}{>} 0$ for each $k \in E$ is equivalent to $\mu \in J(I, E)$ (for any μ such that $\mu_k = 0$ if $k \notin E$). Since any global maximum must satisfy the first-order conditions, we have shown that $\text{Arg max}_{\mu \in \mathbb{R}_+^K | \mu_k = 0 \text{ if } k \notin E} TW(\mu, I) \subseteq J(I, E)$.

In order to establish the reverse inclusion and thus conclude our proof, we do establish next that the function $\mu \rightarrow TW(\mu, I)$ is globally concave. Deriving the equation (16) with respect to μ_l and using (26), we obtain that

⁶³This is the integral of the (interim) probability that a bidder with valuation x wins the object as x varies from X_S to u conditional on z .

$$\begin{aligned}
\frac{\partial^2 TW(\mu, I)}{\partial \mu_k \partial \mu_l} &= \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot \left[V_k^{ent}([N+k]_{+l}, I) - V_k^{ent}(N+k, I) \right] \\
&= \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot \left[\int_{X_S}^{\bar{x}} (F^{(1:N_{+l} \cup I)}(x) - F^{(1:[N+k]_{+l} \cup I)}(x) + F^{(1:N+k \cup I)}(x) - F^{(1:N \cup I)}(x)) dx \right] \\
&= - \sum_{N \in \mathbb{N}^K} P(N|\mu) \cdot E_Z \left[\int_{X_S}^{\bar{x}} \prod_{k=1}^K [F_k(x|Z)]^{n_k} \cdot \prod_{i \in I} F_i^I(x|Z) \cdot (1 - F_l(x|Z))(1 - F_k(x|Z)) dx \right] \\
&= -E_Z \left[\int_{X_S}^{\bar{x}} \prod_{k=1}^K e^{-\mu_k \cdot [F_k(x|Z)]} \cdot \prod_{i \in I} F_i^I(x|Z) \cdot (1 - F_l(x|Z))(1 - F_k(x|Z)) dx \right] \leq 0
\end{aligned} \tag{30}$$

for any $k, l \in \{1, \dots, K\}$. Let \mathbf{H}_I^μ denote the Hessian matrix of the function $\mu \rightarrow TW(\mu, I)$ at the vector of participation rates μ . In order to show that $\mu \rightarrow TW(\mu, I)$ is concave on \mathbb{R}_+^K (for any $I \subseteq \mathcal{I}$), it is sufficient to show that \mathbf{H}_I^μ is negative semi-definite for any μ in \mathbb{R}_+^K (and any $I \subseteq \mathcal{I}$).

Let $Q(x, z) := [(1 - F_1(x|z)), \dots, (1 - F_K(x|z))]$. For $X \in \mathbb{R}^K$, let X^\top its transpose. More generally, the notation $^\top$ is used for any matrix. We then have to show that $X^\top \cdot \mathbf{H}_I^\mu \cdot X \leq 0$ for any $X \in \mathbb{R}^K$ and any $\mu \in \mathbb{R}_+^K$. From (30), we have:

$$X^\top \mathbf{H}_I^\mu X = -E_Z \left[\int_{X_S}^{\bar{x}} \prod_{k=1}^K e^{-\mu_k \cdot [F_k(x|Z)]} \cdot \prod_{i \in I} F_i^I(x|Z) \cdot \underbrace{X^\top \cdot Q(x, Z)^\top Q(x, Z) \cdot X}_{=[Q(x, Z)X]^\top \cdot [Q(x, Z)X] \geq 0} dx \right] \leq 0. \tag{31}$$

In other words, (31) says that \mathbf{H}_I^μ can be viewed as a weighted sum (including integrals) with positive weights of the negative semi-definite matrices $-[Q(x, z)]^\top Q(x, z)$ and is thus also negative semi-definite.

Q.E.D.

Example 3 Consider one incumbent having for sure valuation x_I . Consider two groups of potential entrants. In group 1, all entrants are the high valuation $\bar{x}_E > x_I$. In group 2, valuations are distributed independently across entrants according to the following distribution: With probability $q \in (0, 1)$ (resp. $1 - q$), an entrant has valuation \bar{x}_E ($\underline{x}_E = x_I$). The entry cost denoted by C is assumed to be the same in groups 1 and 2 with $C < q(\bar{x}_E - x_I)$ in order to guarantee entry is profitable. If there are no set-asides, then only bidders from group 1 will enter. The corresponding equilibrium entry rate μ_1 is characterized by

$$e^{-\mu_1}(\bar{x}_E - x_I) = C.$$

On the contrary, if bidders from group 1 are excluded then the equilibrium entry rate of group

2 is now given by $\tilde{\mu}_2$ such that

$$q \cdot e^{-\tilde{\mu}_2 \cdot q} (\bar{x}_E - x_I) = C.$$

Let fix C , \bar{x}_E and x_I such that $\frac{C}{\bar{x}_E - x_I}$ remains constant (strictly below e^{-1} in order to guarantee than $\mu_1 > 1$) while C and $\bar{x}_E - x_I$ go jointly to zero. Then the revenue without exclusion will be approximatively (up to a term that is of the same order as C) equal to $(1 - e^{-\mu_1}) \cdot x_I$ and $(1 - e^{-\tilde{\mu}_2}) \cdot x_I$ if group 1 is excluded. Our aim is now to show that $\tilde{\mu}_2 > \mu_1$ if q is picked adequately. Let $q = 1 - \epsilon$. From equilibrium conditions and then Taylor expansion, we have $\frac{e^{-\mu_1 q}}{e^{-\tilde{\mu}_2 q}} = \frac{qe^{-\mu_1 q}}{e^{-\mu_1}} = 1 + \epsilon \cdot (1 - \mu_1) + o(\epsilon)$. If ϵ is small enough we have then $\tilde{\mu}_2 > \mu_1$. By appropriate choice of the parameters, it is thus profitable to exclude group 1. \diamond

Proof of Lemma 5.1 Consider a dominant-strategy auction and take a given vector of reports $X = (x_1, \dots, x_n)$ for the competing bidders of the given incumbent.⁶⁴ Let $P_X : \mathbb{R}_+ \rightarrow [0, 1]$ denote the function that gives the probability that the given incumbent gets the good as a function of his reported valuation. From Mookherjee and Reichelstein (1992) and taking the perspective of the given incumbent, the function $P_X(\cdot)$ should be non-decreasing and the expected payoff of the given incumbent with type z is equal to $\int_0^z P_X(u) du$ up to a constant, and that constant is null in an auction without participation fees. Furthermore, if the good is over-assigned (resp. under-assigned) to the given incumbent, then the auction assigns the good with probability 1 (resp. 0) to him if it is efficient (inefficient) to do so. Formally, this means that $P_X(z) = 1$ if $z \geq \max\{\max_{i=1, \dots, n} \{x_i\}, X_S\} \equiv x^*$ (resp. $P_X(z) = 0$ if $z \leq x^*$). This further implies that the expected payoff of the incumbent with type z is larger (resp. smaller) than $\max\{z - x^*, 0\}$, i.e. the payoff he will get in the Vickrey auction.

Since the auction assigns the good efficiently among the opponents of the given incumbent, then if the latter wins (resp. does not win) the good, his marginal contribution to the welfare is $z - x^*$ (resp. 0). If the auction over-assigns the good to the given incumbent, then we get that his expected payoff is larger than his marginal contribution to the welfare. If the auction under-assigns the good to the given incumbent, then his expected payoff and his marginal contribution to the welfare are null if he does not get the good $z \leq x^*$. By contrast, if he gets the good (in which case we have necessarily $z \geq x^*$), his expected payoff is smaller than $z - x^*$ and thus smaller than his marginal contribution to the welfare.

The previous arguments hold ex post, namely for any realization of the profile of valuations of the incumbent's competitors. It holds thus a fortiori from an interim perspective, namely for any realization of the set $N \in \mathbb{N}_+$ of competitors but before knowing their valuations. **Q.E.D.**

Proof of eq. (20) Let $TW(\mu, I; \phi(b, r))$ and $\Pi^{inc}(\mu, I; \phi(b, r))$ denote respectively the expected total welfare and the incumbent's expected payoff in a dominant strategy auction without

⁶⁴ X is an empty list if the given bidder faces no competitors.

participation fees that implements a $\phi(b, r)$ -assignment rule and when the entry profile is $\mu \in \mathbb{R}_+^K$ when the (single) incumbent is present. Let $G_\mu(\cdot)$ (resp. $g_\mu(\cdot)$) denote the CDF (resp. PDF) of the highest valuation among the entrants. In a $\phi(b, r)$ -auction and if the incumbent's valuation is distributed independently of the entrants' valuations, the probability that the incumbent wins the auction if his valuation is x is given by $G_\mu(\phi(x))$ if $x \geq r$ and is null if $x < r$. The expected payoff of the incumbent if his type is $x \geq r$ is thus given by

$$\int_r^x G_\mu(b(u))du = (x - r) \cdot G_\mu(X_S) + \int_r^x (x - u) \cdot d[G_\mu(b(u))]. \quad (32)$$

So his expected payoff is the same as in a second-price auction with the reserve price r but if an entrant with valuation x bids as if his valuation were $b^{-1}(x)$. The incumbent's expected payoff (before learning his type) is then $\Pi^{inc}(\mu, I; \phi(b, r)) = \int_r^\infty [\int_r^x G_\mu(b(u))du] \cdot d[F^I(x)] = \int_r^\infty G_\mu(b(x)) \cdot (1 - F^I(x))dx$. With the (single) incumbent, we have by definition of a $\phi(b, r)$ -auction:

$$TW(\mu, I; \phi(b, r)) = F^I(r) \cdot TW(\mu, \emptyset; \phi(b, r)) + \int_r^\infty \left(x \cdot G_\mu(b(x)) + \int_{b(x)}^\infty ud[G_\mu(u)] \right) d[F^I(x)].$$

After an integration per part and plugging the expression of $\Pi^{inc}(\mu, I; \phi(b, r))$, we obtain that

$$\begin{aligned} TW(\mu, I; \phi(b, r)) - TW(\mu, \emptyset; \phi(b, r)) &= \Pi^{inc}(\mu, I; \phi) + \left(r \cdot G_\mu(X_S) + \int_{X_S}^\infty xd[G_\mu(x)] - TW(\mu, \emptyset; \phi(b, r)) \right) \cdot (1 - F^I(r)) \\ &\quad + \int_r^\infty (x - b(x)) \cdot (1 - F^I(x)) \cdot d[G_\mu(b(x))]. \end{aligned}$$

Since $TW(\mu, \emptyset; \phi(b, r)) = TW(\mu, \emptyset) = X_S G_\mu(X_S) + \int_{X_S}^\infty xd[G_\mu(x)]$, we obtain (20). **Q.E.D.**

Proof of Lemma 5.2 Consider a dominant-strategy auction and take a given vector of reports $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ for the opponents of the given bidder i .⁶⁵ Let $P_{x_{-i}} : \mathbb{R}_+ \rightarrow [0, 1]$ denote then the function that gives the probability that bidder i gets the good as a function of his reported valuation (in particular $P_{x_{-i}}(x_i) = \phi_i(X)$). From Mookherjee and Reichelstein (1992) and taking the perspective of the given bidder, the function $P_{x_{-i}}(\cdot)$ should be non-decreasing and the expected payoff of the given bidder with type z is equal to $\int_0^z P_{x_{-i}}(u)du$ up to a constant, and that constant is null in an auction without participation fees. Furthermore, if the good is over-assigned (resp. under-assigned) to bidder i , then the auction assigns the good with probability 1 (resp. 0) to him if it is efficient (inefficient) to do so. Let $\bar{z}_{x_{-i}} = \inf\{z \in \mathbb{R}_+ | \phi_i(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n) = 1\}$ and $\underline{z}_{x_{-i}} = \sup\{z \in$

⁶⁵ X is an empty list if the given bidder faces no competitors.

$\mathbb{R}_+|\phi_i(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n) = 0\}$. Let $\bar{w}_{x_{-i}}$ denote the welfare when bidder i is absent and \bar{w}_X denote the welfare with all bidders. Note that the loser neutral property implies that $\bar{w}_X = P_{x_{-i}}(x_i) \cdot x_i + (1 - P_{x_{-i}}(x_i)) \cdot \bar{w}_{x_{-i}}$ or equivalently the contribution of bidder i to the welfare can be written as $P_{x_{-i}}(x_i) \cdot (x_i - \bar{w}_{x_{-i}})$. Moreover, if the good is over-assigned (resp. under-assigned) to bidder i , then $\bar{z}_{x_{-i}} \leq \bar{w}_{x_{-i}}$ (resp. $\bar{z}_{x_{-i}} \geq \bar{w}_{x_{-i}}$).

Consider first the case where the good is over-assigned to bidder i . We can restrict ourselves to the cases where the contribution of bidder i to the welfare is positive so that $x_i > \bar{z}_{x_{-i}}$ and $P_{x_{-i}}(x_i) = 1$. The contribution to the welfare $x_i - \bar{w}_{x_{-i}}$ is smaller than $x_i - \bar{z}_{x_{-i}}$ which is a lower bound on bidder i 's payoff.

Consider next the case where the good is under-assigned to bidder i . We can restrict ourselves to the cases where bidder i gets the good with positive probability so that $x_i > \bar{z}_{x_{-i}}$. The contribution to the welfare $P_{x_{-i}}(x_i) \cdot (x_i - \bar{w}_{x_{-i}})$ is larger than $P_{x_{-i}}(x_i) \cdot (x_i - \bar{z}_{x_{-i}})$ which is larger than $\int_{\bar{z}_{x_{-i}}}^{x_i} P_{x_{-i}}(u) du$ and then larger than $\int_0^{x_i} P_{x_{-i}}(u) du$, i.e. bidder i 's payoff. **Q.E.D.**

Proof of Theorem 3

With a single group of entrants, we use the simplified notation $n \equiv N$ and $n + 1 \equiv N_{+1}$. Let ϕ denote the assignment rule in the auction.

The assumption that the probability to win the good for an entrant with a given signal does not increase when an extra entrant participates implies that $V^{ent}(n + 1, I; \phi) \leq V^{ent}(n, I; \phi)$ for both $I = \{i\}$ (where i refers to the single incumbent) and $I = \emptyset$ and for any $n \geq 1$. This further implies that $\Pi^{ent}(\mu, I; \phi)$ is nonincreasing in μ . Note also that given that the auction involves no entry fees, then $\mu \cdot \Pi^{ent}(\mu, I; \phi) \leq \bar{x} - \mu \cdot C$. On the whole the set of equilibrium candidates with (resp. without) the incumbent, i.e. if $I = \{i\}$ (resp. $I = \emptyset$), is an interval denoted by $[\underline{\mu}_{in}^*, \bar{\mu}_{in}^*] \subset \mathbb{R}_+$ (resp. $[\underline{\mu}_{out}^*, \bar{\mu}_{out}^*] \subset \mathbb{R}_+$). Note, e.g., that $\underline{\mu}_{in}^* = \bar{\mu}_{in}^* = 0$ if $\Pi^{ent}(0, \{i\}; \phi) < 0$.

We apply Lemma 5.1 for the entrants: If auction ϕ over-assigns the good to the entrants, then $W(n + 1, I; \phi) - W(n, I; \phi) \leq V^{ent}(n + 1, I; \phi)$ for each $n \geq 0$ and thus

$$\frac{\partial TW}{\partial \mu}(\mu, I; \phi) \leq \Pi^{ent}(\mu, I; \phi) \quad (33)$$

for both $I = \{i\}$ and $I = \emptyset$. Note that $\frac{\partial TW(\mu, \emptyset; \phi)}{\partial \mu}|_{\mu=0} = W(1, \emptyset; \phi) - C$. Note that the assumption that the assignment rule ϕ over-assigns to the entrants implies that the difference $W(1, \emptyset) - W(1, \emptyset; \phi)$ is bounded by the probability the entrant has a valuation below X_S times X_S which thus implies that $W(1, \emptyset; \phi) \geq W(1, \emptyset) - X_S$. From the assumption $W(1, \emptyset) > X_S + C$ and using (33), we get that $\Pi^{ent}(0, \emptyset; \phi) > 0$ which implies that $\underline{\mu}_{out}^* > 0$.

The assumption that the probability to win the good for an entrant with a given signal decreases after adding the incumbent implies that $\Pi^{ent}(\mu, \{i\}; \phi) < \Pi^{ent}(\mu, \emptyset; \phi)$ which then implies that $\underline{\mu}_{out}^* > \bar{\mu}_{in}^*$.

Let $\mu_{in}^* \in [\underline{\mu}_{in}^*, \bar{\mu}_{in}^*]$ (resp. $\mu_{out}^* \in [\underline{\mu}_{out}^*, \bar{\mu}_{out}^*]$) denote the equilibrium entry rate with (resp. without) the incumbent. We have thus $\mu_{out}^* > \bar{\mu}_{in}^* \geq \mu_{in}^*$.

Given that $\mu \rightarrow \Pi^{ent}(\mu, \{i\}; \phi)$ is nonincreasing, (33) implies that $\frac{\partial TW(\mu, \{i\}; \phi)}{\partial \mu} < 0$ for any $\mu > \bar{\mu}_{in}^*$. Since $\mu_{out}^* > \bar{\mu}_{in}^* \geq \mu_{in}^*$, we have thus $TW(\mu_{in}^*, \{i\}; \phi) > TW(\mu_{out}^*, \{i\}; \phi)$.

We apply Lemma 5.1 again but now for the incumbent: If auction ϕ under-assigns the good to the (single) incumbent i , then $W(n, \{i\}; \phi) - W(n, \emptyset; \phi) \geq V_i^{inc}(n, \{i\}; \phi)$ for each $n \geq 0$ which further implies that

$$TW(\mu, \{i\}; \phi) - TW(\mu, \emptyset; \phi) \geq \Pi_i^{inc}(\mu, \{i\}; \phi). \quad (34)$$

By reporting a null valuation, the incumbent can guarantee himself a null payoff. Consequently, in a dominant strategy auction we must have $\Pi_i^{inc}(\mu, \{i\}; \phi) \geq 0$ for any $\mu \geq 0$ which thus implies that $TW(\mu_{out}^*, \{i\}; \phi) \geq TW(\mu_{out}^*, \emptyset; \phi)$.

Finally we get that $TW(\mu_{in}^*, \{i\}; \phi) > TW(\mu_{out}^*, \{i\}; \phi)$ which concludes the proof. **Q.E.D.**

Proof of Theorem 4

Let μ_1^*, μ_2^* denote the equilibrium participation profile without set-asides. If $\mu_1^* = 0$, then excluding bidders from group 1 does not have any impact. Suppose then that $\mu_1^* > 0$. Let ϕ denote the assignment rule in the auction.

For $i = 1, 2$, let $\hat{\mu}_i(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ denote the function that maps the equilibrium participation rate of entrants from group i as a function of the participation rate from the other group. Formally, for any $\mu \geq 0$, $\hat{\mu}_1(\mu)$ is characterized by $\Pi_1^{ent}(\hat{\mu}_1(\mu), \mu; \phi) = 0$ (resp. ≤ 0) if $\hat{\mu}_1(\mu) > 0$ (resp. $= 0$). Similarly, the function $\hat{\mu}_2$ is characterized by $\Pi_2^{ent}(\mu, \hat{\mu}_2(\mu); \phi) = 0$ (resp. ≤ 0) if $\hat{\mu}_2(\mu) > 0$ (resp. $= 0$).

If ϕ is loser neutral, then for any vector $X := (x_1, \dots, x_n)$ and any pair of bidders i and j in $\{1, \dots, n\}$, we have $\phi_j(X) \leq \phi_j(X_{-i})$. Consequently, the winning probability of bidder j conditional on any given signal x_j can not decrease if bidder i is excluded. Consequently, when a bidder faces more competitors, then it necessarily reduces its expected payoff. In our two group environment, $V_i^{ent}(N; \phi)$ is nondecreasing in both n_1 and n_2 for $i = 1, 2$. Furthermore, $V_i^{ent}(N_{+i}; \phi) < V_i^{ent}(N; \phi)$ once $V_i^{ent}(N; \phi) > 0$, for $i = 1, 2$.⁶⁶ We obtain then that $\Pi_k^{ent}(\mu_1, \mu_2; \phi)$ is nondecreasing in both μ_1 and μ_2 and furthermore than $\frac{\partial \Pi_k^{ent}(\mu_1, \mu_2; \phi)}{\partial \mu_k} < 0$ if $\Pi_k^{ent}(\mu_1, \mu_2; \phi) > -C_k$.

It is then straightforward that $\mu \rightarrow \hat{\mu}_1(\mu)$ and $\mu \rightarrow \hat{\mu}_2(\mu)$ are both non-increasing and continuous. If $\hat{\mu}_2(0) = 0$, then group 2 is always inactive and it is thus straightforward that it is detrimental to exclude bidders from group 1. We assume next that $\hat{\mu}_2(0) > 0$.

If the profile $(0, \hat{\mu}_2(0))$ satisfies the equilibrium condition without set-asides, then it corresponds to the equilibrium profile if bidders from group 1 are excluded and our equilibrium selection assumption thus guarantees that it is detrimental to exclude group 1. Suppose then that $(0, \hat{\mu}_2(0))$ is not an equilibrium profile (without set-asides). This implies that $\hat{\mu}_1(\hat{\mu}_2(0)) > 0$.

⁶⁶If a bidder from group i has a strictly positive probability to win the good so does his extra competitor from group i so that his probability to win the good (strictly) decreases with extra competitors from group i .

Consider the function $\mu \rightarrow \widehat{\mu}_1(\widehat{\mu}_2(\mu))$. The group 1 equilibrium profile candidates (without set-asides) corresponds to a fixed point of this function, or equivalently (given that $(0, \widehat{\mu}_2(0))$ is not an equilibrium profile) the set of μ such that $\Pi_1^{ent}(\widehat{\mu}_1(\widehat{\mu}_2(\mu)), \widehat{\mu}_2(\mu); \phi) = 0$. Consider the smallest solution (as a fixed point) which we denote by $\underline{\mu}_1 > 0$. Since the function $\mu \rightarrow \widehat{\mu}_1(\widehat{\mu}_2(\mu))$ is continuous, we have then $\widehat{\mu}_1(\widehat{\mu}_2(\mu)) \geq \mu$ for any $\mu \in [0, \underline{\mu}_1]$ and thus $\Pi_1^{ent}(\mu, \widehat{\mu}_2(\mu); \phi) \geq 0$ for any $\mu \leq \widehat{\mu}_1(\widehat{\mu}_2(\mu))$.

Let $\widetilde{TW}(\mu; \phi) := TW((\mu, \mu_2^*(\mu)); \phi)$ denote the net expected welfare as a function of the entry rate from group 1 given that group 2 entry rate is the equilibrium one in the auction ϕ . We show next that $\widetilde{TW}(\underline{\mu}_1; \phi) \geq \widetilde{TW}(0; \phi)$. By our equilibrium selection assumption and given that the seller's expected revenue coincides with the total welfare, if this inequality is true then it holds for the equilibrium profile which will allow us to conclude.

To show this inequality, write:

$$\frac{d\widetilde{TW}(\mu; \phi)}{d\mu} = \frac{\partial TW}{\partial \mu_1}(\mu, \widehat{\mu}_2(\mu); \phi) + \underbrace{\frac{d\widehat{\mu}_2(\mu)}{d\mu}}_{\leq 0} \cdot \frac{\partial TW}{\partial \mu_2}(\mu, \widehat{\mu}_2(\mu); \phi).$$

Then we apply Lemma 5.2: If auction ϕ under-assigns the good to bidders from group 1, then

$$W(N_{+1}; \phi) - W(N; \phi) \geq V_1^{ent}(N_{+1}; \phi) \quad (35)$$

and thus $\frac{\partial TW}{\partial \mu_1}(\mu_1, \mu_2; \phi) \geq \Pi_1^{ent}(\mu_1, \mu_2; \phi)$. Hence, we obtain that $\frac{\partial TW}{\partial \mu_1}(\mu, \widehat{\mu}_2(\mu); \phi) \geq \Pi_1^{ent}(\mu, \widehat{\mu}_2(\mu); \phi) \geq \Pi_1^{ent}(\widehat{\mu}_1(\widehat{\mu}_2(\mu)), \widehat{\mu}_2(\mu); \phi) = 0$ for any $\mu \in [0, \underline{\mu}_1]$.

If auction ϕ over-assigns the good to bidders from group 2, then

$$W(N_{+2}; \phi) - W(N; \phi) \leq V_2^{ent}(N_{+2}; \phi) \quad (36)$$

and thus $\frac{\partial TW}{\partial \mu_2}(\mu_1, \mu_2; \phi) \leq \Pi_2^{ent}(\mu_1, \mu_2; \phi)$. Hence, we obtain that $\frac{\partial TW}{\partial \mu_2}(\mu, \widehat{\mu}_2(\mu); \phi) \leq \Pi_2^{ent}(\mu, \widehat{\mu}_2(\mu); \phi) = 0$.

Gathering the previous inequalities, we obtain that $\frac{d\widetilde{TW}(\mu; \phi)}{d\mu} \geq 0$ for any $\mu \in [0, \underline{\mu}_1]$, and thus $\widetilde{TW}(\underline{\mu}_1; \phi) \geq \widetilde{TW}(0; \phi)$, as needed to complete the proof. **Q.E.D.**

Technical computations for Section 6.2

Proof of Propositions 6.2 and 6.3

We proceed by differentiating the revenue when the strength of incumbent i increases. Let $G_\lambda(\cdot) = [F_j(\cdot)]^\lambda$. The limit case $\lambda = 0$ corresponds to the case where incumbent i has been excluded while $\lambda = 1$ corresponds to the case where he is not excluded. Note that we have $\frac{dG_\lambda(x)}{d\lambda} = \log[F_j(x)] \cdot G_\lambda(\cdot) \leq 0$.

Next we put the argument λ in the payoff and revenue functions in order to indicate that the strength of the incumbent i is parameterized by λ . With respect to our previous notation, we will have in particular $\Pi^{ent}(\mu, I; \lambda) = \Pi^{ent}(\mu, I)$ (resp. $= \Pi^{ent}(\mu, I_{-i})$) if $\lambda = 1$ (resp. $\lambda = 0$) and

similarly for any $j \in I_{-i}$, $\Pi_j^{inc}(\mu, I; \lambda) = \Pi_j^{inc}(\mu, I)$ (resp. $= \Pi_j^{inc}(\mu, I_{-i})$) if $\lambda = 1$ (resp. $\lambda = 0$).

The equilibrium entry rate $\mu(\lambda)$ as a function of the strength λ of the special incumbent i is characterized (provided that this is some entry) by the equation:

$$\Pi^{ent}(\mu(\lambda), I; \lambda) = \sum_{n=0}^{\infty} e^{-\mu(\lambda)} \frac{[\mu(\lambda)]^n}{n!} \cdot \int_{X_S}^{\infty} \prod_{j \in I_{-i}} F_j^I(x) G_\lambda(x) [F(x)]^n \cdot (1 - F(x)) dx = C$$

or equivalently

$$\int_{X_S}^{\infty} \prod_{j \in I_{-i}} F_j^I(x) G_\lambda(x) e^{-\mu(\lambda)[1-F(x)]} \cdot (1 - F(x)) dx = C.$$

We have then

$$\frac{d\mu(\lambda)}{d\lambda} = \frac{\int_{X_S}^{\infty} \prod_{j \in I_{-i}} F_j^I(x) \frac{dG_\lambda(x)}{d\lambda} e^{-\mu(\lambda)[1-F(x)]} \cdot (1 - F(x)) dx}{\int_{X_S}^{\infty} \prod_{j \in I_{-i}} F_j^I(x) G_\lambda(x) e^{-\mu(\lambda)[1-F(x)]} \cdot (1 - F(x))^2 dx}.$$

Similarly, the rent of an incumbent $j^* \in I_{-i}$ as a function of λ is given by

$$\Pi_{j^*}^{inc}(\mu(\lambda), I; \lambda) = \int_{X_S}^{\infty} \prod_{j \in I_{-i} \setminus \{j^*\}} F_j^I(x) G_\lambda(x) e^{-\mu(\lambda)[1-F(x)]} \cdot (1 - F_{j^*}^I(x)) dx = C.$$

After differentiating the rent of such an incumbent rents with respect to λ , we get:

$$\begin{aligned} \frac{d\Pi_{j^*}^{inc}(\mu(\lambda), I; \lambda)}{d\lambda} &= \frac{\partial \Pi_{j^*}^{inc}(\mu(\lambda), I; \lambda)}{\partial \lambda} + \frac{d\mu(\lambda)}{d\lambda} \cdot \frac{\partial \Pi_{j^*}^{inc}(\mu(\lambda), I; \lambda)}{\partial \mu} \\ &= \int_{X_S}^{\infty} \prod_{j \in I_{-i}} F_j^I(x) \frac{dG_\lambda(x)}{d\lambda} e^{-\mu(\lambda)[1-F(x)]} \cdot (1 - F(x)) dx \\ &\quad \left[\frac{\int_{X_S}^{\infty} \prod_{j \in I_{-i} \setminus \{j^*\}} F_j^I(x) \frac{dG_\lambda(x)}{d\lambda} e^{-\mu(\lambda)[1-F(x)]} \cdot (1 - F_{j^*}^I(x)) dx}{\int_{X_S}^{\infty} \prod_{j \in I_{-i}} F_j^I(x) \frac{dG_\lambda(x)}{d\lambda} e^{-\mu(\lambda)[1-F(x)]} \cdot (1 - F(x)) dx} - \frac{\int_{X_S}^{\infty} \prod_{j \in I_{-i} \setminus \{j^*\}} F_j^I(x) G_\lambda(x) e^{-\mu(\lambda)[1-F(x)]} \cdot (1 - F_{j^*}^I(x)) \cdot (1 - F(x)) dx}{\int_{X_S}^{\infty} \prod_{j \in I_{-i}} F_j^I(x) G_\lambda(x) e^{-\mu(\lambda)[1-F(x)]} \cdot (1 - F(x))^2 dx} \right]. \end{aligned}$$

Let $U_j(x) := \frac{1}{F_j^I(x)} \cdot \frac{1 - F_j^I(x)}{1 - F(x)}$, which has been assumed to be decreasing on (\underline{x}, \bar{x}) for any $j \in I_{-i}$.

Let h_1 denote the density function defined by $h_1(x) := \frac{\prod_{j \in I_{-i}} F_j^I(x) \left| \frac{dG_\lambda(x)}{d\lambda} \right| e^{-\mu(\lambda)[1-F(x)]} \cdot (1 - F(x))}{\int_{X_S}^{\infty} \prod_{j \in I_{-i}} F_j^I(y) \left| \frac{dG_\lambda(y)}{d\lambda} \right| e^{-\mu(\lambda)[1-F(y)]} \cdot (1 - F(y)) dy}$ for $x \geq X_S$ and 0 otherwise.

Let h_2 denote the density function defined by $h_2(x) := \frac{\prod_{j \in I_{-i}} F_j^I(x) G_\lambda(x) e^{-\mu(\lambda)[1-F(x)]} \cdot (1 - F(x))^2}{\int_{X_S}^{\infty} \prod_{j \in I_{-i}} F_j^I(y) G_\lambda(y) e^{-\mu(\lambda)[1-F(y)]} \cdot (1 - F(y))^2 dy}$ for $x \geq X_S$ and 0 otherwise.

Note that the support of the densities h_1 and h_2 is $[\max\{X_S, \underline{x}\}, \bar{x}]$.

We have then

$$\frac{d\Pi_{j^*}^{inc}(\mu(\lambda), I; \lambda)}{d\lambda} = - \int_{X_S}^{\infty} \prod_{j \in I_{-i}} F_j^I(x) \left| \frac{dG_\lambda(x)}{d\lambda} \right| e^{-\mu(\lambda)[1-F(x)]} \cdot (1 - F(x)) dx \cdot \left[E_{x \sim h_1}[U_{j^*}(x)] - E_{x \sim h_2}[U_{j^*}(x)] \right].$$

Note also that the assumption that $x \rightarrow -\frac{\log[F_j^I(x)]}{1-F(x)}$ is decreasing on $[\underline{x}, \bar{x}]$ implies that the likelihood ratio $\frac{h_1}{h_2}$ is decreasing on the interior of its support. Likelihood ratio dominance implies first-order stochastic dominance (see Appendix B in Krishna (2003)). Since U_i is decreasing on (\underline{x}, \bar{x}) , we obtain finally that $E_{x \sim h_1}[U_{j^*}(x)] > E_{x \sim h_2}[U_{j^*}(x)]$ and thus that $\frac{d\Pi_{j^*}^{inc}(\mu(\lambda), I; \lambda)}{d\lambda} < 0$ which further implies by integration that $\Pi_{j^*}^{inc}(\mu(1), I; 1) < \Pi_{j^*}^{inc}(\mu(0), I; 0)$ for any $j^* \in I_{-j}$. This concludes the proof of Proposition 6.2.

By Taylor expansion of the expression (22) with respect to the parameter λ , the impact on the revenue to exclude the incumbent i is of the form:

$$\frac{dR(\mu(\lambda), I)}{d\lambda} \Big|_{\lambda=0} = \underbrace{\frac{\partial TW(\mu(\lambda), I_{-i})}{\partial \mu} \Big|_{\lambda=0}}_{=0} \cdot \underbrace{\frac{d\mu(\lambda)}{d\lambda} \Big|_{\lambda=0}}_{\text{well defined}} + \sum_{j \in I_{-i}} \frac{d\Pi_j^{inc}(\mu(\lambda), I; \lambda)}{d\lambda}$$

The latter terms in the sum are all strictly negative. The revenue is thus locally decreasing in λ around the origin (namely when λ is close to zero) and we obtain thus Proposition 6.3.

A sufficient condition for Assumption 2

Next lemma provides a sufficient simple condition on the CDFs of the incumbents that will guarantee that Assumption 2 holds. A special case of it is when incumbents and entrants are all symmetric, i.e. when $F_i^I = F$ for any incumbent i .

Lemma A1 Let $G(x) := \sum_{j=1}^{\infty} s_j \cdot [F(x)]^j$ where $s_j \in [0, 1]$ for any j and $\sum_{j=1}^{\infty} s_j = 1$. Assume that the support of F is $[\underline{x}, \bar{x}]$. Then the functions $x \rightarrow \frac{1-G(x)}{G(x)(1-F(x))}$ and $x \rightarrow \frac{-\log[G(x)]}{(1-F(x))}$ are both decreasing on (\underline{x}, \bar{x}) .

Proof

On (\underline{x}, \bar{x}) , we have

$$\frac{(1-G(x))}{G(x)(1-F(x))} = \frac{1}{G(x)} \cdot \sum_{j=1}^{\infty} s_j \cdot \frac{1-[F(x)]^j}{1-F(x)} = \frac{\sum_{j=1}^{\infty} s_j \cdot \sum_{k=0}^{j-1} [F(x)]^k}{\sum_{j=1}^{\infty} s_j \cdot [F(x)]^j}. \quad (37)$$

Taking the derivative of the right-hand side of (37), we obtain that the derivative of $\frac{(1-G(x))}{G(x)(1-F(x))}$ has the same sign as $(\sum_{j=2}^{\infty} s_j \cdot \sum_{k=1}^{j-1} k[F(x)]^{k-1}) \cdot (\sum_{j=1}^{\infty} s_j \cdot [F(x)]^j) - (\sum_{j=1}^{\infty} s_j \cdot \sum_{k=0}^{j-1} [F(x)]^k) \cdot (\sum_{j=1}^{\infty} s_j \cdot j[F(x)]^{j-1})$.

In order to obtain that $\frac{(1-G(x))}{G(x)(1-F(x))}$ is strictly decreasing on (\underline{x}, \bar{x}) , it is thus sufficient to check that

$$\frac{\sum_{j=1}^{\infty} s_j \cdot j[F(x)]^{j-1}}{\sum_{j=1}^{\infty} s_j \cdot [F(x)]^j} > \frac{\sum_{j=2}^{\infty} s_j \cdot \sum_{k=1}^{j-1} k[F(x)]^{k-1}}{\sum_{j=1}^{\infty} s_j \cdot \sum_{k=0}^{j-1} [F(x)]^k}$$

if $F(x) > 0$. This comes from the fact that $\frac{j[F(x)]^{j-1}}{[F(x)]^j} > \frac{\sum_{k=1}^{j-1} k[F(x)]^{k-1}}{\sum_{k=0}^{j-1} [F(x)]^k}$ for any $j \geq 2$, which is equivalent to the inequalities $\sum_{k=0}^{j-1} j \cdot [F(x)]^{k-1} > \sum_{k=1}^{j-1} k \cdot [F(x)]^{k-1}$ for any $j \geq 2$, which obviously hold.

We consider now the function $x \rightarrow \frac{-\log[G(x)]}{(1-F(x))}$. On (\underline{x}, \bar{x}) , its derivative is given by

$$\frac{\partial\left[\frac{-\log[G(x)]}{(1-F(x))}\right]}{\partial x} = \frac{f(x)}{(1-F(x))^2} \cdot \left[-\frac{\sum_{j=1}^{\infty} s_j \cdot (1-F(x))j \cdot [F(x)]^{j-1}}{G(x)} - \log[G(x)] \right]$$

From the inequality $\log(x) > -\frac{1-x}{x}$ for any $x \in (0, 1)$, we have

$$G(x) \cdot \log[G(x)] > -(1-G(x)) = \sum_{j=1}^{\infty} s_j \cdot (1-F(x)) \sum_{k=0}^{j-1} [F(x)]^k \geq \sum_{j=1}^{\infty} s_j \cdot (1-F(x))j \cdot [F(x)]^{j-1}$$

On the whole, we obtain that $\frac{\partial\left[\frac{-\log[G(x)]}{(1-F(x))}\right]}{\partial x} < 0$ on (\underline{x}, \bar{x}) .

Q.E.D.

Proof of Proposition 7.3

What remains to be shown is the result 3. When entrants are symmetric (so that μ reduces to a scalar), the equilibrium entry profile without subsidies, denoted by μ_{no-fee}^* , satisfies $\mu_{no-fee}^* \in \max_{\mu \geq 0} \{TW(\mu, I)\}$. Let us assume that $\mu_{no-fee}^* > 0$ (which implies in particular that entrants have the highest valuations with positive probability). Since $\mu \rightarrow TW(\mu, I)$ is (strictly) concave and the functions $\mu \rightarrow \Pi_j^{inc}(\mu, I)$, for $j \in I$, are nonincreasing, we obtain that the equilibrium entry profile with the optimal subsidy, denoted by $\mu_{with-fee}^*$ and which is characterized by $\mu_{with-fee}^* \in \max_{\mu \geq 0} \{TW(\mu, I) - \sum_{j \in I} \Pi_j^{inc}(\mu, I)\}$, satisfies $\mu_{with-fee}^* \geq \mu_{no-fee}^*$. Since $\mu \rightarrow \Pi^{ent}(\mu, I)$ is nonincreasing with $\lim_{\mu \rightarrow \infty} \Pi^{ent}(\mu, I) = -C$, we have thus $-C < \Pi^{ent}(\mu_{with-fee}^*, I) \leq \Pi^{ent}(\mu_{no-fee}^*, I) = 0$ which means that the optimal fee takes the form of a partial reimbursement of the entry cost C . **Q.E.D.**