Group decision making

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VERY PRELIMINARY

Abstract

1 Introduction

We use a simple model of collective search to examine group decision making. Groups are composed of sub-groups. Support for a proposal may vary across members and across sub-groups. Simple decision rules only take into account the number of members that support a proposal. Complex decision rules also take into account support within sub-groups. We examine the efficiency of these various rules as a function of the heterogeneity of preferences within and across sub-groups.

2 Basic Model

We consider a group consisting of n members, labeled $i = 1, ..., n$. At any date $t = 1, ...,$ if a decision has not been made yet, a new proposal is drawn

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and examined. A proposal is denoted u. The set of proposals is denoted U. If the proposal u is implemented, it gives member i utility u_i. The utilities $(u_i)_{i=1}^n$ of the proposal u may vary from one proposal to the next. Each u_i belongs to $[\underline{u}, \overline{u}]$, and we assume that proposals at the various dates $t = 1, ..$ are drawn *independently* from the same distribution with continuous density $f(\cdot) \in \Delta([\underline{u}, \overline{u}]^n)$.

Upon arrival of a new proposal u, each member decides whether to accept that proposal. The game stops whenever the current proposal receives sufficiently strong support. We call the rule that specifies whether support is strong enough *a decision rule*. A simple decision rule specifies whether the proposal is accepted as a function of the total number of members that supports it: under the m-majority rule, the game stops whenever at least m out of the n members vote in favor of the proposal.

We shall also consider more complex rules in which acceptance of a proposal may depend on (i) the total number of members that accept it (ii) the number of members in each subgroup that accept it (iii) the number of subgroups that accept it.

In its form general form, a decision rule is formalized as follows. A vote or decision to support for member i is denoted $z_i = 0, 1$ where $z_i = 1$ stands for support and $z_i = 0$ for no support, and a decision rule is a mapping $\rho(.)$ from the vector of individual votes $z = (z_i)_i$ to $\{0, 1\}$, where $\rho(z) = 1$ stands for support.

We normalize to 0 the payoff that parties obtain under perpetual disagreement, and we let δ denote the common discount factor of the committee members. That is, if the proposal u is accepted at date t, the date 0 payoff of member i is $\delta^t u_i$.

Strategies and equilibrium. In principle, a strategy specifies an acceptance rule that may at each date be any function of the history of the game. We will however restrict our attention to *stationary* equilibria of this game, where each member adopts the same acceptance rule at all dates.[1]

Given any stationary acceptance rule σ_{-i} followed by members j, $j \neq i$, we may define the expected payoff $\bar{v}_i(\sigma_{-i})$ that member i derives given σ_{-i} from following his (best) strategy. An optimal acceptance rule for member i is thus to accept the proposal u if and only if

$$u_i \geq \delta \bar{v}_i(\sigma_{-i}),$$

which is stationary as well (this defines the best-response of member i to σ_{-i}).

Stationary equilibrium acceptance rules are thus characterized by a vector $v = (v_1, .., v_n)$ such that member i votes in favor of u if $u_i \geq \delta v_i$ and votes against it otherwise. For any decision rule $\rho(.)$ and value vector v, it will be convenient to refer to $A_{v,\rho}$ as the corresponding *acceptance set*, that is, the set of proposals that get support given that each i supports u if and only if $u_i \geq \delta v_i$:

$$A_{v,\rho} = \{u \in U, \text{ for } z_i = 1_{u_i \geq \delta v_i} \text{ and } z = (z_i)_i, \; \rho(z) = 1\}. \tag{1}$$

Equilibrium consistency then requires that

$$v_i = \Pr(u \in A_{v,\rho}) E[u_i \mid u \in A_{v,\rho}] + [1 - \Pr(u \in A_{v,\rho})] \, \delta v_i \tag{2}$$

or equivalently

$$v_i = \frac{\Pr(u \in A_{v,\rho})}{1 - \delta + \delta \Pr(u \in A_{v,\rho})} E[u_i \mid u \in A_{v,\rho}]. \tag{3}$$

A stationary equilibrium is characterized by a vector v and an acceptance set $A_{v,\rho}$ that satisfy (1)-(2). It always exists, as shown in Compte and Jehiel (2004-09).

[1] To avoid coordination problems that are common in voting (for example, all players always voting "no"), we will also restrict attention to equilibria that employ no weakly dominated strategies (in the stage game). These coordination problems could alternatively be avoided by assuming that votes are sequential.

Preferences within and across subgroups.

We now specialize our general framework. We assume that there are K subgroups in the population, labelled $k = 1, ..., K$. Proposals affect subgroups differently, and there is also some heterogeneity in each subgroup.

Formally, a proposal is characterized by a vector $x \in (x_1, ..., x_K) \in X$, and proposals are drawn from X according to some density g. The utility that some individual i in subgroup k derives is

$$u_i^{(k)} = x_k + \varepsilon_i$$

where ε_i is assumed to be a random variable independant of x. We shall denote by F_k its cumulative density. We shall assume that F_k has a density f_k that is single peaked and without loss of generality we also assume that $E\varepsilon_i = 0$. We shall also denote by $\bar{\eta}$ an upperbound on the support of ε_i.

For each subgroup, two features of interest will be *subgroup homogeneity* (which depends on how concentrated the density f_k is) and *skewdness* (summarized by $F_k(0)$, i.e. the fraction of individuals in subgroup k that get a below average payoff – as compared to own subgroup).

Finally, we denote by α_k the fraction of individuals that belong to subgroup k. Throught out the paper we shall be interested in the set of equilibrium values that obtains in limit case where the group size n is arbitrarily large. We shall refer to such values as *limit equilibrium values.*

3 An inefficiency result

There may be two types of inefficiencies. Inefficiencies due to delays in reaching agreement. Inefficiencies resulting from agreement on a pareto inferior outcome. In this Section, we focus on simple decision rules (i.e. rules where agreement depends on the total number of individuals that accept it). We denote by β the fraction of individuals that is required for a proposal to be accepted and refer to this rule as a $\beta-$majority rule. We show that there may be no $\beta-$majority rules that generates efficient decision making.

We consider two subgroups ($K = 2$) of equal size.

> **Proposition:** Assume that $F_k(0) > 1/2$ for each k. Then, there exists $\bar{\delta}$ such that for all $\delta > \bar{\delta}$, and for any $\beta-$majority rule, limit equilibrium values remain bounded away from the Pareto frontier of X.

Consider n large and denote by v_k the expected equilibrium value for an individual in group k. Consider now any draw $x = (x_1, x_2)$. The individuals in group k that accept x are those for which

$$x_k + \varepsilon_i \geq \delta v_k.$$

For a large n, there is thus a fraction approximately equal to $1-F_k(\delta v_k - x_k)$ that accept it. The set of proposals that pass, which we denote by A, is thus:

$$A = \{(x_1, x_2), \sum_k \alpha_k(1 - F_k(\delta v_k - x_k)) > \beta\}$$

For almost efficient decision making, the set A should be $\nu-$close to the equilibrium vector v, for some ν close to 0. This thus requires, for δ close enough to 1

$$\sum_k \alpha_k(1 - F_k(\nu)) > \beta$$

hence, choosing ν small so that $1 - F_k(\nu) < \bar{\beta} < 1/2$ for each k,

$$\beta < \max_k 1 - F_k(\nu) < \bar{\beta}$$

But now observe that $\alpha_k > \bar{\beta}$ for all $k = 1, 2$. So for any candidate (v_1, v_2), the set A contains all (x_1, x_2) such that $x_k > v_k + \bar{\eta}$ for some k, hence it contains draws that are far away from the efficient frontier.Q.E.D.

Intuitively, there may be two forms of inefficiencies. The first form typically arises when the majority requirement is small: then each individual has a low acceptance threshold because there is a high chance that he will not be part of the majority that accepts, and as a result, the agreement set is large and include Pareto inferio outcomes. The second form arises when the majority requirement is too large: then each individual sees little risk that the outcome will hurt him, and he prefers to patiently wait for a nice draw, and as a result, inefficient delays arise.

The logic of the argument in the proof of the Proposition is that for a majority rule $\beta < \min \alpha_k$, it is sufficient that a subgroup unanimuously agrees to a proposal to pass it. As a consequence, whatever the candidate equilibrium values (v_1, v_2), the agreement set must be large, and the first type of inefficiencies applies. Now for majority rules $\beta > \max 1 - F_k(0)$, draws (x_1, x_2) close to the candidate equilibrium values (v_1, v_2) cannot pass because they do not get enough support: only draws that are strictly more efficient than (v_1, v_2) may pass, which requires that (v_1, v_2) is bounded away from the frontier (inefficient delays must arise in that case).

Under the assumptions of the proposition, $\max 1 - F_k(0) < \min \alpha_k$ so inefficiencies must arise whatever the majority rule.

4 The effect of size and heterogenity

We examine below how the size of a subgroup as well as its homogeneity affect its strength. We still consider the case of two subgroups and focus on the case where the distributions ε_i are centered and where a single subgroup cannot on its own enforce a proposal. We will further assume that X is symmetric so that differences in expected payoffs only stem from asymmetries in size or differences in the distributions F_k.

Formally, we assume:

> **A1:** f_k is centered on 0 for each k, and $\beta > \max \alpha_k$. Besides X is symmetric

Now for any $\lambda < 1$, we define the set

$$B(\lambda) = \{(x_1, x_2), \sum_k \alpha_k(1 - F((\lambda - 1)x_k)) \geq \beta\}$$

To interpret $B(\lambda)$, observe that under A1, the majority requirement is strong enough that the set of accepted proposals will turn out be a small set, concentrated around some proposal (x_1, x_2) on the frontier of X, and that the equilibrium outcome will involve inefficient delays. Equilibrium values will thus satisfy $v_k = \lambda x_k$ for some $\lambda < 1$, and the term $\sum_k \alpha_k(1 - F((\lambda - 1)x_k))$ thus corresponds to the fraction of individuals that accept (x_1, x_2). A proposal $x \in B(\lambda) \cap X$ is thus a good candidate for constructing an equilibrium. But this is not good enough, because if $B(\lambda) \cap X$ is not a singleton, then other proposals will get accepted.

Accordingly, in what follows, we define λ^* as the highest value of λ such that $B(\lambda) \cap X \neq \emptyset$, and denote by $x^* = (x_1^*, x_2^*)$ that point of intersection.[2]

We have the following Proposition.

Proposition 2: Limit equilibrium values satisfy $v^* = \lambda^* x^*$. Besides, in equilibrium, only proposals close to x^* get accepted.

Assuming that the Pareto frontier of X is parameterized by

$$g(x) \equiv 0,$$

and since at the solution x^*, the sets $B(\lambda^*)$ and X have the same tangent, an immediate corollary of Proposition 2 is the following:

Corollary: *Limit equilibrium values satisfy $v^* = \lambda^* x^*$ where the solution (x^*, λ^*) is characterized by the equations:*

$$g(x^*) = 0 \tag{4}$$

[2] λ^* and x^* are uniquely defined because X is a convex set and because under A1, $B(\lambda)$ is also a convex set (also recall that the distributions f_k are single peaked).

$$\sum_k \alpha_k(1 - F_k((\lambda^* - 1)x_k^*)) = \beta \quad (5)$$

$$\frac{g_1'(x^*)}{g_2'(x^*)} = \frac{\alpha_1 f_1((\lambda^* - 1)x_1^*)}{\alpha_2 f_2((\lambda^* - 1)x_2^*)} \quad (6)$$

We now wish to use the above Corollary to illustrate the effect of size and homogeneity.

Throughout the rest of this Section, we assume that the Pareto frontier of X is parameterized by $g(x) = 0$, and to fix ideas, we set:

$$g(x) = (x_1)^a + (x_2)^a - 1, \text{ with } a > 1$$

We also assume that densities take the form:

$$f_k(\varepsilon) = \frac{1}{b_k}(1 - \frac{\mid \varepsilon \mid}{b_k}), \text{ with } b_k > 0,$$

where b_k is thus a measure of the dispersion of the preferences within subgroup k.

Set $y_k = (1 - \lambda^*)x_i^*$. With this change of variable, and using Equations (4) and (6), we are looking for (y_1, y_2) such that

$$(\frac{y_1}{y_2})^{a-1} = \frac{\alpha_1}{\alpha_2}\frac{f_1(-y_1)}{f_2(-y_2)} \quad (7)$$

$$\alpha_1 F_1(-y_1) + \alpha_2 F_2(-y_2) = 1 - \beta \quad (8)$$

We can then derive λ^* from

$$1 - \lambda^* = ((y_1)^a + (y_2)^a)^{1/a}$$

Assume $b_1 > b_2$. If $\alpha_1 < \alpha_2 \frac{b_1}{b_2}$, then we can define $y^* > 0$ such that

$$\alpha_1 f_1(-y^*) = \alpha_2 f_2(-y^*),$$

We then let

$$\beta^* = \sum_k \alpha_k(1 - F_k(-y^*)).$$

Otherwise, note that we have $\alpha_1 f_1(-y) > \alpha_2 f_2(-y)$ for all $y \in (-b_2, b_2)$.

We have the following Proposition:

Proposition 3: *Assume* $b_1 > b_2$*. If* $\alpha_1 > \frac{b_1}{b_2}\alpha_2$*, then* $x_1^* > x_2^*$*.*
If $\alpha_1 < \frac{b_1}{b_2}\alpha_2$*, then*
(i) at majority rule β^**,* $x_1^* = x_2^* = x_0^*$*, and* $\lambda^* = \lambda_0$*, where* $x_0^* \equiv (1/2)^{1/a}$ and $\lambda_0 \equiv 1 - y^*/x_0^*$.
(ii) for any more stringent majority rule $\beta > \beta^*$*, we have:*

$$x_1^* > x_0^* > x_2^* \text{ and } \lambda^* < \lambda_0,$$

(iii) for any less stringent majority rule $\beta < \beta^*$*, we have*

$$x_1^* < x_0^* < x_2^* \text{ and } \lambda^* > \lambda_0$$

Proof: Consider $\beta > \beta^*$, and assume by contradiction that $y_1 < y_2$. Then (7) implies $\alpha_1 f_1(-y_1) < \alpha_2 f_2(-y_2)$. Since $0 < y_1 < y_2$, we have $\alpha_1 f_1(-y_2) < \alpha_1 f_1(-y_1)$, hence $\alpha_1 f_1(-y_2) < \alpha_2 f_2(-y_2)$, which requires $y_2 < y^*$, hence

$$y_1 < y_2 < y^*.$$

(8) then implies:

$$1 - \beta = \alpha_1 F_1(-y_1) + \alpha_2 F_2(-y_2) > \sum_k \alpha_k F_k(-y^*) = 1 - \beta^*.$$

Contradiction. So $y_1 > y_2$, hence by a similar argument, $y_1 > y_2 > y^*$; Also note that we must thus have $x_1^* > x_2^*$, hence $x_1^* > (1/2)^{1/a} > x_2^*$, hence $y^* = (1 - \lambda_0^*)(1/2)^{1/a} < y_2 = (1 - \lambda^*)x_2^* < (1 - \lambda^*)(1/2)^{1/a}$, which implies that $\lambda^* < \lambda_0^*$. This proves claim (ii). Claim (iii) is proved similarly.

To prove claim (i), observe that $y_1 < y_2$ implies $\alpha_1 f_1(-y_2) < \alpha_2 f_2(-y_2)$ which cannot happen when $\alpha_1 > \frac{b_1}{b_2}\alpha_2$. Q.E.D.

The following figures plot the ratio v_1/v_2 and λ^* as a function of the majority rule β for specific values of the parameters α_k and b_k.

5 Designing an efficient rule.

In this Section, we show that efficiency may be restored when one considers decision rules that take into account within subgroup support. Specifically, we examine rules where support by subgroup k obtains when a fraction β_k of its members supports it, and where a proposal is adopted when all subgroups support it. The rules that we now examine are thus *subgroup based rules* that require enough support within subgroups, and unanimity across subgroups. Formally, they are characterized by a vector $(\beta_1, ..., \beta_K)$ where K is the number of subgroups.

Alternatively, under rule $(\beta_1, ..., \beta_K)$, a proposal may be vetoed by subgroup k if and only if there is a fraction $1 - \beta_k$ that of its members that opposes the proposal.

We show below that there always exist a sub-group based rule that induces approximate efficiency.

> **Proposition 4**: *For any* $\xi > 0$, there exists $\eta > 0$ *such that the sub-group based rule* $(\beta_1, ..., \beta_K)$ *where* $\beta_k = 1 - F_k(\eta)$ *generates* $\xi-$*efficient decisions when individuals are patient enough.*

Consider n large and denote by v_k the expected equilibrium value for an individual in group k. Consider now any draw $x = (x_1, x_2)$. The individuals in group k that accept x are those for which

$$x_k + \varepsilon_i \geq \delta v_k.$$

For a large n, there is thus a fraction approximately equal to $1-F_k(\delta v_k - x_k)$ that accept it. Subgroup k supports proposal x with probability arbitrarily close to 1 (as n gets large) when $1 - F_k(\delta v_k - x_k) > \beta_k = 1 - F_k(\eta)$ or equivalently when $x_k > \delta v_k - \eta$, and arbitrarily close to 0 (as n gets large) when $x_k < \delta v_k - \eta$. The set of proposals that pass, which we denote by A_v, is thus:

$$A_v = \{x = (x_1, ..., x_K) \in X,\ x_k > \delta v_k - \eta,\ \text{for all } k\}$$

Note now for any candidate equilibrium vector $v \in X$, $\Pr A_v$ is bounded away from 0, so when players are patient enough, v_k should be close to $E[x_k \mid x \in A_v]$.

Now assume by contradiction that v were away from the frontier by more than $\xi = \eta^{1/2}$, then we would have $E[x_k \mid x \in A_v] > v_k + O(\xi - \eta)$, so for η small enough, v_k could not be close to $E[x_k \mid x \in A_v]$. Contradiction. Q.E.D.

6 When a constitutional agreement is lacking

The previous Section has show that efficiency could be restored when the decision rule takes within subgroup support into account and each subgroup has veto rights. The decision rule that restores efficiency however requires that subgroups agree on what constitutes within subgroup support. We show in this Section that when such an agreement is lacking and each subgroup is free to decide upon what constitute appropriate support, then inefficiencies arise again: each subgroup is tempted to increase its majority requirement β_k. Such an increase tilts the outcome in a way that is favorable to members of subgroup k, but it generates efficiency losses in terms of delays.

In what follows, we assume that in a first stage, subgroups simultaneously choose the majority requirement β_k that applies within their own group, and that in a second stage, our previous collective search game is played. The following Proposition derives the equilibrium outcome of this two stage game. For simplicity, we make the following assumption:

> **A2**: X is the simplex: $X = \{x = (x_1, ..., x_K), \sum_{k \in K} x_k \leq 1, x_k \geq 0 \text{ for all } k\}$.

We first derive the continuation equilibrium outcome in the subgame where each subgroup k has chosen β_k respectively. It will be convenient to let

$$\gamma_k \equiv -F_k^{-1}(1 - \beta_k).$$

Consider a candidate equilibrium value (v_1, v_2). Consider any draw $x = (x_1, x_2)$. Following the proof of Proposition 4, subgroup k supports proposal x with probability arbitrarily close to 1 (as n gets large) when $1 - F_k(\delta v_k - x_k) > \beta_k$ or equivalently when

$$x_k > \delta v_k + \gamma_k$$

The agreement set thus coincides with:

$$A_v = \{x \in X, \quad x_k > \delta v_k + \gamma_k \text{ for } k = 1, 2\}$$

Let $\lambda_v = \frac{\Pr A_v}{1-\delta+\delta \Pr A_v}$. In equilibrium, it should be that

$$v_k = \lambda_v E[x_k \mid x \in A_v]$$

If $\gamma_k > 0$ for some k, then, for patient individuals, we must have $\lambda_v < 1$ and $\Pr A_v$ must be comparable to $1 - \delta$, hence, for patient individuals, we must have $E[x_k \mid x \in A_v] \simeq v_k + \gamma_k$.

If, for some other k', $\gamma_{k'} \leq 0$, then we must have $v_{k'} \leq \lambda_v v_{k'}$, so $v_{k'} = 0$.

If $\gamma_k > 0$ for each k, we thus get

$$\frac{v_k}{v_1} = \frac{v_k + \gamma_k}{v_1 + \gamma_1}$$

hence

$$\frac{v_k}{v_1} = \frac{\gamma_k}{\gamma_1}.$$

Now if X is the simplex, we have:

$$\lambda_v = \sum_k v_k = \frac{v_1}{\gamma_1}(\sum_k \gamma_k) \text{ and } \lambda_v = \frac{v_1}{v_1 + \gamma_1}$$

hence,

$$v_1 = \frac{\gamma_1}{\sum_k \gamma_k} - \gamma_1$$

Intuitively, a more stringent majority rule for subgroup 1 tilts the outcome more favorably to that subgroup, but it induces costs in terms of inefficient delays.

We deduce the following proposition:

Proposition 5: *The two stage game has a unique pure strategy equilibrium. Each subgroup chooses β_k so that $\gamma_k = \gamma^* = \frac{K-1}{K^2}$, and the resulting equilibrium payoffs are $v_k = \frac{1}{K} - \gamma^* = \frac{1}{K^2}$.*

Note that the equilibrium outcome does not depend on the distributions F_k. It depends however on the shape of X. Assuming that the frontier of X can be parameterized by $g_a(x) = 0$ with

$$g_a(x) = \sum_k (x_k)^a - K^{1-a}$$

for some $a > 1$, the following figure plots the sum of expected payoffs as a function of a. As a increases, transferability is reduced and the efficiency loss is reduced.