Bargaining over Randomly Generated Offers: A new perspective on multi-party bargaining

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October 2004

Abstract

We study a $n-$person bargaining problem where offers are generated randomly and each party decides to accept or reject the current proposal. Bargaining terminates when n_0 out of the n agents accept the current proposal. The effect of patience, the number of players, as well as the majority requirement (n_0) are examined. For example, as n_0 increases, we find the following tradeoff: more efficient outcome are obtained, but it takes more time to reach them. We also relate the solutions obtained to some classic results in bargaining and voting.

1 Introduction

A central feature of Rubinstein's alternating offer bargaining game is the assumption that each party in turn makes a take it or leave it offer to the other party, and that each party has full control over which offer he makes to the other party. In equilibrium, a party whose turn it is to move has a

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strategic advantage, that he exploits by choosing an offer that makes the responding party indifferent between accepting the offer now and getting that strategic advantage for herself next period (thus at the cost of delaying the agreement).

In practice however, the proposals put on the table are often the outcome of a complex process, that both parties try to influence, but over which neither party has full control. Besides, preferences may be subject to random shocks. In such circumstances, even a player who would have full control over offers made in the physical space would find it difficult to target a specific utility level for the other parties.

A first objective of the paper is to provide a model of bargaining where parties no longer have full control over the offers made along the bargaining process. In the basic version of the model, proposals will be drawn at random, and negotiation will continue until an agreement is reached, that is (in case unanimous consent is required), until all parties accept the same proposal. In effect, we are thus transforming the standard bargaining model into a simple timing game in which each party just decides when to say "yes".[1]

A second motivation for our work is to provide a new and *simple* model of multi-party bargaining. Many folk ideas in multi-person bargaining do not yet, to our knowledge, have theoretical support. For example, it is frequently suggested that bargaining with many parties should be more difficult than bargaining with few parties. Also, it would seem that agreement is more difficult to obtain when agreement requires unanimous consent, than when a smaller majority requirement prevails. No theory comes in support of these

[1]Another way to think of our model is in terms of a multi-person search process governed by a pre-agreed termination procedure (i.e. a majority requirement to enforce the choice of a proposal).

intuitions however.[2,3] We believe that our model with random proposals outlined above may be a good step towards a better understanding of these issues.

The paper is organized as follows. We present the model in Section 2, and analyze the trade-off between various majority requirements in Section 3. One strength of our model is its simplicity. This allows us to deal with standard set-ups, but also with less standard ones. Section 4 and 5 show that despite its simplicity, our model generates the usual solutions in the classic set-ups. And it provides novel insights in the less standard ones. Finally, in Section 6, we analyze some extensions of our model to the case where parties have some control or influence over the proposals put on the table.

2 The Model

There are n parties, labeled $i = 1, ..., n$. At any date $t = 1, ...$, if an agreement has not been reached yet, parties receive a proposal for agreement. The set of proposals will vary in a space of dimension $m \geq 1$. We denote by x a proposal, by X the set of proposals, and we assume that X is isomorphic to $[0, 1]^m$. We also assume that proposals at the various dates $t = 1, ..$ are drawn *independently* from the same distribution with continuous density $f \in \Delta(X)$. So players have no influence on the proposals that get to the negotiation table. In Section 6, we will amend the model and analyze the case where parties may affect the distribution of proposals.

[2]Theory might even provide support for the opposite intuition: A core allocation exists under unanimity rule, while it does not under, say, majority rule.

[3]A recent and interesting exception is Cai (2000, 2002) who considers settings where one active bargainer negotiates *sequentially* with N passive bargainers. (Inefficient delays occur in these models because bargaining is sequential, and each passive player prefers to be the last one to negotiate with the active player). Such results are in the vein of the hold-up literature.

At any date, after a proposal x has been made, each party (sequentially)[4] decides whether to accept the proposal. Under the unanimity rule, the game stops whenever all parties accept the current proposal. Under the majority rule, or any qualified majority rule, the game stops whenever, say, n_0 out of the n agents decide to accept.[5]

We let $u_i(x)$ denote the utility that player i derives at date t from an agreement x obtained at t. We assume that $u_i(.)$ is continuous, and we normalize to 0 the payoff that parties obtain in the absence of agreement. Let X_0 denote the set of proposals that are individually rational for all players:

$$X_0 = \{x \in X, u_i(x) > 0 \text{ for all } i\}$$

Throughout the paper, we will assume that X_0 is non-empty.

We assume that payoffs are discounted with a common discount factor δ, so that viewed from date 1, agreement on x obtained at date t yields party i a discounted payoff equal to $\delta^{t-1}u_i(x)$.

In principle, a strategy specifies an acceptance rule that may at each date be any function of the history of the game. We will however restrict our attention to *stationary* equilibria of this game, where each party adopts the same acceptance rule at all dates.

Given any stationary acceptance rule σ_{-i} followed by other parties, we may define the largest expected payoff $\bar{v}_i(\sigma_{-i})$ that player i may derive given σ_{-i} from following his (best) strategy. An optimal acceptance rule for party i is thus to accept the proposal x if and only if

$$u_i(x) \geq \delta \bar{v}_i(\sigma_{-i}),$$

which is stationary as well (this defines the best-response of player i to σ_{-i}).

[4]We assume the responses are made sequentially to rule out the weakly dominated strategies that may be used in such coordination problems. An alternative is to assume that players do not use weakly dominated strategies.

[5]More generally, a termination procedure could be described as a general function of individual acceptance decisions.

Under unanimous consent, stationary acceptance rules may thus be characterized by a vector $v = (v_1, .., v_n)$ that satisfies, in equilibrium:

$$v_i = PE[u_i(x) \mid u_j(x) \geq \delta v_j \ \forall j] + (1 - P)\delta v_i \tag{1}$$

where $P = \Pr\{u_j(x) \geq \delta v_j \ \forall j\}$, or equivalently

$$v_i = \rho E[u_i(x) \mid u_j(x) \geq \delta v_j \ \forall j] \tag{2}$$

where

$$\rho = \frac{P}{1 - \delta + \delta P}$$

It will be convenient to refer to A as the acceptance set. Under the unanimity rule, this acceptance is defined by:

$$A = \{x \in X, u_i(x) \geq \delta v_i \forall i\}.$$

Under more general termination procedures, such as the qualified majority rule (n_0), the acceptance set becomes:[6]

$$A = \{x \in X, \exists N_0 \subset \{1, ..., n\}, \ \mid N_0 \mid = n_0, \ u_i(x) \geq \delta v_i \forall i \in N_0\},$$

and equilibrium values satisfy

$$v_i = PE[u_i(x) \mid x \in A] + (1 - P)\delta v_i \tag{3}$$

where $P = \Pr(x \in A)$.

One of our objective will be to compare equilibrium values under various majority requirements. Before turning to the analysis of the model, we first derive an existence result:

> **Proposition 1:** *Whatever the majority requirement (n_0), a stationary equilibrium exists.*

Proof: Define the function $v \to \phi(v)$, where $\phi_i(v)$ coincides with the RHS of Equation (3), and let $\bar{u} = \max_{i,x} u_i(x)$. The function ϕ is continuous from $[0, \bar{u}]^n$ to itself, hence it has a fixed point. **Q. E. D.**

[6] For any finite set B, $\mid B \mid$ denotes the cardinality of B.

3 Unanimity versus Qualified majority.

We start with an application of our model to multi-party bargaining. Our objective is to analyze the trade-off between the unanimity rule and the various possible majority requirements. Our model will give substance to the idea that more stringent majority requirements improve the efficiency of the agreement reached, but at the cost of more delays in finding an agreement (because finding a proposal that satisfies more people takes more time).

In order to quantify this trade-off, we make specific assumptions in this Section concerning preferences and the distribution over proposals. Specifically, we examine the case of a surplus of size S to be shared, assuming transferable utilities and a distribution over offers that is uniform on the simplex, $X = \{x \in R_n^+, \sum_i x_i \leq S\}$. We will later return to more general assumptions.

3.1 Inefficiencies under unanimous consent.

Bargaining under unanimous consent will induce inefficiencies, essentially because parties have to wait for a proposal that satisfies everybody. Intuitively, as the number of players grows, it should be more difficult to get a draw that satisfies *all* players, and the resulting inefficiency will therefore grow with the number of parties. We will confirm that intuition in the next Proposition in the case of patient parties.

Before that note that with arbitrarily patient players, bargaining will be close to efficient. Indeed, assume by contradiction that the equilibrium value vector remains away from the Pareto frontier by some amount Δ even as discounting gets close to 1. Then, for any equilibrium value v, the acceptance set $A = \{x \in X, u_i(x) \geq \delta v_i\}$ would be large and the probability of agreement P significant (i.e. comparable to Δ^n), which means that agreement arises relatively fast (with probability close to 1, it will occur before a date small compared to $1/(1-\delta)$). In particular, agreement will continue to arise fast even if, say player 1 raises his acceptance level by $\Delta/2$. Hence this would be a profitable deviation, because he would obtain $\delta v_1 + \Delta/2 >> v_1$ relatively fast. Even though the outcome must be close to efficient, there are still some inefficiencies left. The following Proposition quantifies how the inefficiency grows with the number of participants:

> **Proposition 2**: *Assume $u_i(x) = x_i$ and that there is a surplus S to be shared. Let Δ_n be the efficiency loss, and let $r(n) = (1-\delta)^{1/n}$. (a) If the distribution over offers is uniform on the simplex $X = \{x \in R_n^+, \sum_i x_i \leq S\}$, then, for δ close to 1, we have:*
>
> $$\Delta_n/S = (\frac{n+1}{n}(1-\delta))^{1/(n+1)} + o(r(n+1)))$$
>
> *(b) If the distribution over offers is uniform on the (n-1-dimensional) simplex $X = \{x \in R_n^+, \sum_i x_i = S\}$, then, for δ close to 1, we have:*
>
> $$\Delta_n/S = r(n) + o(r(n))$$
>
> *Besides, in both cases, in any period the probability of agreement is comparable to $(1-\delta)/\Delta_n$.*

Proposition 2 implies that as the number of parties grows, the inefficiency grows substantially. When $1-\delta = 1/10000$, unanimous consent among 4 participants already implies an efficiency loss comparable to 10%. Note that inefficiencies would remain substantial, even if offers were more "localized", that is closer to the offer $(S/n, ..., S/n)$. Assume that offers fall in the simplex $X_\Delta = \{x \in R_n^+, \sum_i x_i \leq S,\, x_i \geq S(1-\Delta)/n\}$. Then it is easy to check that Δ_n/S is reduced by a factor Δ.

Proposition 2 also shows that, not surprisingly, when offers may be inefficient, the efficiency loss increases further. The reason is that conditional on satisfying all players, the offer now involves an efficiency loss (comparable to $n/(n+1)$).

Proof: Because f is symmetric, there exists a symmetric equilibrium characterized by v. Let $V = nv$ be the sum of equilibrium values, and $s = \sum_i x_i$. In case (a), we have.

$$P = \Pr\{x_i \geq \delta v\} = (1 - \delta V/S)^n$$

and[7]

$$V = \rho E[s \mid x_i \geq \delta v \ \forall i] = \rho(\delta V + \frac{n}{n+1}(S - \delta V))$$

with $\rho = P/(1-\delta+P)$. Ignoring the terms of order higher than 1 in $(1-\delta)$, these equalities imply that

$$\frac{n+1}{n}(1-\delta)V/S = (1 - V/S)^{n+1}.$$

which further implies the desired statement.

In case (b), we have

$$P(v) = (1 - \delta V/S)^{n-1}$$

and

$$V = \rho S$$

which implies, ignoring the terms of order higher than 1 in $(1 - \delta)$,

$$(1 - \delta)V/S = (1 - V/S)^n$$

which implies the desired statement. **Q. E. D.**

3.2 Inefficiencies under qualified majority rule.

Now how different is the situation when only a qualified majority of $n_0 < n$ participant is required? Then the set of realizations for which agreement will occur is much larger, and most importantly, its measure does not tend

[7] Note that since f is uniform, $E[s \mid s \leq s_0] = \frac{n}{n+1}s_0$. (This is because the volume of the $(n-1)$-dimensional simplex, $\sum_i x_i = s$ is equal to as^{n-1} for some constant a).

to 0 when players become arbitrarily patient. So agreement will obtain much faster. To see why intuitively, observe that parties cannot expect more than S/n on average. So they will accept any proposal that gives that amount. Let

$$A_0 = \{x \in X, \exists N_0 \subset N, \ \mid N_0 \mid = n_0, \, x_i \geq S/n \text{ for all } i \in N_0\}$$

In any period, the probability of agreement is at least equal to $\Pr A_0$. The set A_0 contains the simplex

$$B = \{x \in X, x_i \geq S/n \text{ for } i = 1, ..., n_0, \text{ and } x_i \geq 0, \text{ for } i = n_0 + 1, ..., n\}.$$

Compared to X, B is just a smaller simplex, with edges of length $S(1-n_0/n)$ instead of S. So its volume if just the fraction $(1 - n_0/n)^n$ the volume of X. Hence the probability of agreement never vanish as δ tends to 1.

So there is a benefit to the qualified majority rule. But of course there is also a potential cost, because agreement may occur even if this is at the expense of the $n - n_0$ parties: n_0 players may agree even if the $n - n_0$ remaining parties gets 0 in that agreement. So the outcome may lie far away from the Pareto frontier. The next Proposition examines this trade-off.

Proposition 3: *Assume* $u_i(x) = x_i$ *and that there is a surplus* S *to be shared. Let* Δ_n *be the efficiency loss. (a) If the distribution over offers is uniform on the simplex* $X = \{x \in R_n^+, \sum_i x_i \leq S\}$, *then, for* δ *close to* 1, *we have:*

$$\Delta_n/S = \frac{1 - n_0/n}{n + 1 - n_0/n} + O((1 - \delta))$$

(b) If the distribution over offers is uniform on the (n-1-dimentional) simplex $X = \{x \in R_n^+, \sum_i x_i = S\}$, *then, for* δ *close to* 1, *we have:*

$$\Delta_n/S = O(1 - \delta)$$

Besides, in both cases, there exists $a > 0$ such that *the probability of agreement in any period exceeds* a *independently of* δ.

So case (b) corresponds to a case where there are no loss to less stringent majority requirements, because proposals are always efficient. The only effect is to speed up agreement, which improves overall efficient compared to the unanimity rule. In case (a) however, there is a cost, which increases as the majority requirement n_0 decreases. This cost however should be compared with that obtained under the unanimity rule, which is comparable to $(1-\delta)^{1/(n+1)}$

Proof: Without loss of generality, we set $S = 1$. We first compute

$$G(z) = E(x_i \mid x_j \geq 1/n - z \text{ for } n_0 \text{ players}).$$

Assume player i's expected valuation is $1/n - z$. Under qualified majority rule, player i has a chance n_0/n of being in the (qualified) majority, and a chance $1 - n_0/n$ of not being in the majority. By symmetry, players get $1/n - z + h$ when they are in the majority, and h when they are not, where $nh = \frac{n}{n+1}(1 - \frac{n_0}{n} + n_0 z)$. It follows that

$$G(z) = \frac{n_0}{n}(\frac{1}{n} - z) + h = \frac{n_0}{n}(\frac{1}{n} - z) + \frac{1}{n+1}(1 - \frac{n_0}{n} + n_0 z)$$

Now choose z so that $\delta v_i = 1/n - z$. We use equilibrium conditions to derive nz, which corresponds to the efficiency loss. We have:

$$\frac{1/n - z}{\delta} = \frac{P}{1 - \delta + \delta P} G(z) \tag{4}$$

Since P is bounded away from 0, we get, forgetting terms of order 1 in $(1-\delta)$:

$$(1 - \frac{n_0}{n})(\frac{1}{n} - \frac{1}{n+1}) = z((1 - \frac{n_0}{n} + \frac{n_0}{n+1})$$

hence

$$nz = \frac{1 - n_0/n}{(n+1) - n_0/n},$$

Note that the probability of agreement in any period is at least[8]

$$Q_{n0} = (1 - \frac{n_0}{n} + n_0 z)^n = (1 - \frac{n_0}{n + 1 - n_0/n})^n$$

In case (b), $G(z) = 1/n$. So Equation (4) yields:

$$1 - nz = \frac{\delta P}{1 - \delta + \delta P}$$

or equivalently, $nz = (1 - \delta)/(1 - \delta + \delta P)$, where P is at least equal to $(1 - n_0/n)^n$. **Q. E. D.**

4 Bargaining under unanimous consent.

One virtue of our model is its tractability. This allows us to deal with classic set-ups in the bargaining and the voting or political science literature. But it also allows us to deal with new issues not addressed in the literature. In the next two Sections, we show that in the classic set-ups, our model provides predictions that coincide with the standard ones, and we also provide predictions for less standard set-ups.

Our model has two frictions: discounting, and the lack of control over offers. We view the second friction as an important ingredient of many negotiation processes. It is a friction commonly assumed in search theory, but it is an unexplored one in the bargaining context. We also view the first one as being important, in particular in light of Section 3, as patience

[8] An exact expression is given by inductive formula:

$$P_{n_0} = \begin{pmatrix} n_0 \\ n \end{pmatrix} Q_{n_0}(z) - P_{n_0+1}$$

where $Q_k(z) = (1 - \frac{k}{n} + kz)^n$, hence

$$P_{n_0} = \sum_{k=0}^{n-n_0} (-1)^k \begin{pmatrix} k \\ n \end{pmatrix} Q_k(z)$$

may turn out to be crucial in determining which majority requirements promote overall efficiency. In the next sections however, we will concentrate on the second friction, and provide predictions of our model assuming that discounting tends to 1.

We start with bargaining set-ups, where we impose the unanimous consent requirement. We examine first the case where the space of proposals is rich, e.g. the dimensionality of the offer space is at least as large as the number of players.

4.1 The case of a rich space of proposals

By rich space of proposals, we mean that local variations in the space of proposals generate all possible variations in the utility space. So in particular, the dimension of the space of proposals (m) must be at least equal to the number of parties (n). We will show that when discounting gets close to 1, equilibrium outcomes must get close to the generalized $n-$person Nash solution (hence to the Rubinstein (1982)'s solution as well in the two person case).[9]

Formally, we make the following assumption, which not only ensures that the space of proposals is rich, but also ensures that the Nash solution is uniquely defined, and that it is not a degenerate point of the Pareto frontier:

Assumption 1: Assume that (i) $u(.)$ is *smooth, (ii)* $u(X)$ is a smooth $n$dimensional convex set, and (iii) $x^* = \arg\max_{x \in X} \prod_i u_i(x)$ is not a boundary point of the Pareto-frontier of $u(X)$.

We have:

> **Proposition 4:** *Assume that Assumption 1 holds, and that f is bounded away from 0 and smooth on X. When δ tends to 1,*

[9]This can be viewed as the analog of Binmore et al.(1986) in our random offer bargaining setup. Note tha we allow for more than two players (but yet restrict attention to stationary equilibria).

equilibrium values tend to v^* such that $v_i^* = u_i(x^*)$. Besides there exist constants $m, M > 0$ such that for any δ and any stationary equilibrium, the probability of agreement P at any date and the equilibrium value v satisfy:

$$m(1-\delta)^{1-1/(n+1)} < P < M(1-\delta)^{1-1/(n+1)} \qquad (5)$$

$$m(1-\delta)^{1/n+1} < \mid v^* - v \mid < M(1-\delta)^{1/n+1} \qquad (6)$$

The intuition as to why as δ tends to 1, equilibrium values must tend to the Pareto frontier has already be explained in Section 3. (If equilibrium values are far away from the Pareto frontier, each player could strictly improve her payoff by tightening her acceptance rule, thus yielding a contradiction.) As δ tends to 1, equilibrium values cannot tend to the Pareto frontier too fast either. This was already apparent in Proposition 3, which dealt with the transferable utility case. The intuition is simple. If v were an equilibrium value close to the frontier, (away from the frontier by $\Delta << (1-\delta)^{1/n}$), then the acceptance set A would be small, the probability of agreement would be small as well (i.e. $P(v) << 1-\delta$), implying that players expected payoff cannot lie too close to the frontier. Thus, yielding a contradiction. It follows that equilibrium values lie away from the frontier by a distance at least of the order of $(1-\delta)^{1/n}$. In Appendix, we will compute more precise bounds on equilibrium values and acceptance probabilities.

Now observe that since f is bounded away from 0 and smooth on X, conditional on acceptance, the distribution over proposals gets close to being uniform on the acceptance set when δ is close to one (because as δ tends to 1, the acceptance set becomes a small set, on which variations of f become tiny). This is essentially why as δ tends to one, the solution is independent of the distribution f.

Let us now characterize the limit equilibrium vector, and show that it coincides with the Nash solution. First observe that equation (2) implies

$$\frac{v_i}{v_1} = \frac{E(u_i(x) - \delta v_i \mid u_j(x) \geq \delta v_j \text{ for all } j)}{E(u_1(x) - \delta v_1 \mid u_j(x) \geq \delta v_j \text{ for all } j)}. \tag{7}$$

Since u is smooth, the distribution over proposals induces a distribution over joint utilities $(u_1, ..; u_n)$ that is close to the uniform distribution on the set

$$D_\delta = \{u \in u(X), u_i \geq \delta v_i\}.$$

Let $g(u) = 0$ be a parameterization of the frontier. Since D_δ is large compared to $(1-\delta)$, it is close to D_1, and since $u(X)$ is a smooth $n-$dimensional convex set, D_1 is itself close to the simplex

$$D = \{u, u_i \geq v_i, \sum a_i(u_i - \bar{v}_i) \leq 0\},$$

where $\bar{v}$ is the point on the frontier closest from v, and where $a_i = g_i'(\bar{v})$.[10]

On the right-hand side of (7), both the numerator and the denominator are comparable to $(1-\delta)^{1/n}$, which is large compared to $(1-\delta)$, hence ignoring terms comparable to $(1-\delta)$ or small compared to $(1-\delta)^{1/n}$, we finally have:

$$\frac{v_i}{v_1} = \frac{E_{unif}(u_i - v_i \mid u \in D)}{E_{unif}(u_1 - v_1 \mid u \in D)}, \tag{8}$$

which implies:

$$\frac{v_i}{v_1} = \frac{a_1}{a_i}$$

We thus obtain the same equations as those obtained from the Nash maximization program.

Asymmetric waiting costs.

In our model, waiting costs are the same for all players. Our analysis can easily be extended to the case of asymmetric waiting costs. The following Proposition establishes the connection with the generalized Nash solution in this case. We assume that each party i has a discount factor δ_i, such that $(1-\delta_i) = (1-\delta)/\alpha_i$.

[10] D_δ has a size comparable to $(1-\delta)^{1/n+1}$. By close, we mean that the frontiers of these sets are away from each other by a term no larger than $(1-\delta)^{2/n+1}$.

Assumption 2: Assume that (i) $u(.)$ is *smooth, (ii)* $u(X)$ is a smooth n-dimensional convex set, and (iii) $v^{**} = \arg\max_{u \in u(X)} \prod_i (u_i)^{\alpha_i}$ is not a boundary point of the Pareto-frontier of $u(X)$.

We have:

Proposition 5: *Assume that Assumption 2 holds, and that f is bounded away from 0 and smooth on X. When δ tends to 1, equilibrium values tend to v^{**}.*

Intuitively, it is not surprising that more patient players get a higher expected payoff in equilibrium. For example assume that all parties have a fixed discount factors δ_i, and consider the case where party 1 would become arbitrarily patient. It is easy to see that as δ_1 approaches one, party 1 must get all the surplus. Indeed, as δ_1 approaches 1, it cannot be that in equilibrium the probability of agreement $P = P(v)$ remains bounded away from 0, because otherwise, party 1 would prefer to demand (substantially) more, at the risk of decreasing the probability of agreement to, say $P/2$. This would increase for him the expected value of agreement, conditional on agreeing. This increase would be at the cost of more delays, but since his waiting costs are small, the deviation would be profitable. And since P must tend to 0, it must be that (i) the equilibrium outcome is close to the Pareto frontier, and (ii) other players equilibrium payoffs get close to 0.

Proof: The analysis is very similar to that of Proposition 4, so details are omitted. Just note that the left hand side of the analog for the case of asymmetric discount factors of Equations (7) and (8) becomes $\frac{(1-\delta_i)v_i}{(1-\delta_1)v_1}$, and that the right hand side of the analog of (8) is unchanged because the acceptance set is large compared to all δ_i, $i = 1, ..., n$. **Q. E. D.**

4.2 Bargaining on a smaller set of alternatives.

In the previous subsection, we have assumed that the set of alternatives was quite rich: local changes in the alternative picked could generate all possible variations in the utility space. When utility is not transferable and bargaining takes place over physical alternatives (other than money), there are a number of applications in which local variations in the alternative picked need not generate all possible variations in the utility space. How does our analysis change in that case?

Our main insight is that the equilibrium outcome may be entirely determined by a subset of agents (despite the fact that all agents have a veto power).

To illustrate the claim, we examine the case where the set of feasible agreement is one-dimensional,[11] and where preferences over these agreements are single-peaked. More specifically, we assume:

Assumption 3: *Assume* $X = [0, 1]$, $0 \leq \theta_1 < ... < \theta_n \leq 1$, $u_i(x) = v(x_i - \theta_i)$, *where* v is smooth, single peaked with a maximum at 0, and positive on $[-1, 1]$.

We will refer to the parameter θ_i as party i's bliss point.

We start by observing that for patient individuals, our model remains predictive: as parties get very patient, the set of accepted proposals becomes arbitrarily close to some limit proposal x^*, and this set is large enough so that delays in reaching agreement induce only negligible costs: a proposal gets accepted in a laps of time short compared to $1/(1-\delta)$. The intuition is very similar to that given earlier. Roughly, consider the set A of accepted agreements. (i) The set A must be small: if it were not small, then agent i would always prefers to veto agreements that would lie furthest away from his bliss point; and (ii) the set A cannot be too small, that is, too concen-

[11] Allowing for more dimensions (but still staying away from the case of rich set of alternatives) would not alter the insight that only a subset of agents drive the equilibrium outcome.

trated around some x^*: otherwise agreement takes time to occur, parties cannot expect more than $\lambda u_i(x^*)$ for some $\lambda < 1$ and the set of proposals that would be accepted unanimously would be large, contradicting the premise that it is small.

In comparison with Section 4.1 however, the locus of the accepted agreements will no longer coincide with the generalized $n-$person Nash solution. Rather, assuming that $v(.)$ is a concave function, it will in general be determined by the characteristics of only two individuals.

> **Proposition 6**: *Let preferences satisfy Assumption 3 with v concave.*[12] *Then when δ tends to 1, only proposals close to some θ^* are accepted. The position θ^* is determined only by the bliss points θ_1 and θ_n of individual 1 and n, and it corresponds to the (Nash) solution obtained in case only individuals 1 and n are present. If v is symmetric around 0, then $\theta^* = \frac{\theta_1+\theta_n}{2}$.*

The solution is thus determined by the Nash bargaining solution among the two individuals with most extreme preferences. This is in contrast with the generalized Nash solution, in which all parties preferences would matter.

Intuitively, equilibrium conditions impose constraints on the set of proposals that may be accepted. When the set of possible proposals is one dimensional and v is concave, the set A of accepted proposals is an interval. This set is thus determined by two conditions, and it is thus not surprising that the preferences of only two players matter.

The reason why only the extremists matter is as follows. Let $A = [\underline{x}, \bar{x}]$ denote the set of outcomes accepted when only individuals 1 and n are present. $\underline{x}$ and $\bar{x}$ are determined so that (i) individual 1 is just indifferent between accepting $\bar{x}$ now and waiting for the arrival of another proposal in A - these proposals are better for him, but he has to wait -, and (ii) individual

[12] *Concavity can be interpreted as reflecting the risk aversion of agents.*

n is just indifferent between accepting $\underline{x}$ now and waiting for the arrival of another proposal in A. Now assume that there are other individual present, with $\theta_i \in (\theta_1, \theta_n)$ for each of them. Say, $\theta_i < \underline{x}$. Because v is concave, and because say θ_i is closer to A than θ_1, individual i cares less about which alternative in A is picked. So given that only alternatives in A can be picked, he does not want to delay further the outcome, and he is willing to accept any proposal in A.

It is interesting to contrast our solution with that which would obtain in a more standard random proposer model (Binmore (1987)), where each party is selected with probability $1/n$ to make an offer (over which he has full control). In such a model, the equilibrium value vector would not coincide with the generalized Nash solution, but it would not coincide with our model either, because the way bliss points are distributed over the segment $[0, 1]$ would now matter. Typically, an equilibrium would consist of a pair $\{\underline{x}, \bar{x}\}$ of proposals: parties with bliss point below $\underline{x}$ would offer $\underline{x}$, and parties with bliss point above $\bar{x}$ would offer $\bar{x}$. The relative frequency with which $\underline{x}$ and $\bar{x}$ are proposed thus depends on the number of parties with bliss points below and above $\underline{x}$ and $\bar{x}$; and so is the locus of $\underline{x}$ and $\bar{x}$: As δ tends to 1, the solution would tend to the weighted Nash solution among the two most extreme individuals (θ_1 and θ_n), in which weights are determined endogenously by the distribution of agents along the segment (θ_1, θ_n).

Proof of Proposition 6: Because v is concave, the set of outcomes accepted by each individual is an interval. So the joint acceptance set is also an interval, say $A = [\underline{x}, \bar{x}]$.

Now let v_i denote player i's equilibrium value. We have:

$$v_i = \Pr(A) E[u_i(x) \mid x \in A] + (1 - \Pr A)\delta v_i,$$

hence

$$\delta v_i = \lambda E[u_i(x) \mid x \in A]$$

where $\lambda = \frac{\delta \Pr A}{1-\delta+\delta \Pr A} < 1$. Define $g(\theta, x) = v(x-\theta) - \lambda E[v(x-\theta) \mid x \in A]$.

Equilibrium conditions therefore require that for all $x \in A$, and for all i, $g(\theta_i, x) \geq 0$.

We wish to show that $g(\theta_1, \bar{x}) = 0$. Indeed, assume $g(\theta_1, \bar{x}) > 0$. Since v is concave, for any $\theta < \bar{x}$, we have

$$\begin{aligned} \frac{\partial g}{\partial \theta}(\theta, \bar{x}) &= -v'(\bar{x} - \theta) + \lambda E[v'(x - \theta) \mid x \in A] \\ &\geq -(1 - \lambda)v'(\bar{x} - \theta) > 0. \end{aligned}$$

Besides, for any $\theta \geq \bar{x}$, $g(\theta, \bar{x}) > 0$. It would thus follow that $g(\theta_i, \bar{x}) > 0$ for all i, hence proposals slightly larger than $\bar{x}$ would be accepted unanimously as well. Similarly, we may show that $g(\theta_n, \underline{x}) = 0$. The interval A is thus solely determined by the preferences of player 1 and n. **Q. E. D.**

5 Bargaining under (qualified) majority requirements.

We now move to settings where unanimous consent is not required.

5.1 The simple majority rule and the median voter prediction

We examine first the majority rule case. Under the assumption that the set of feasible alternative is one-dimensional, and that preferences over these alternatives are single peaked, the standard voting model is quite predictive when there is an odd number of agents: a unique outcome turns out to be stable,[13] the one preferred by the median voter. What we will show below is that the same prediction obtains in our case.[14]

[13] An outcome is stable if there is no alternative outcome that a majority would prefer.

[14] This insight should be compared with that of Baron (1996), who also obtains that same prediction, using the bargaining model of Baron and Ferejohn (1989) (based on the random proposer model of Binmore (1987)). For further work on the relationship between Baron and Ferejohn's model and the core, see Banks and Duggan (2000).

Our model is rather different from a standard voting model: it is dynamic, and agents have the choice between voting for one particular alternative (randomly selected), or remaining at the status quo. Of course, voting against the proposed alternative does not imply that agents will remain at the status quo for ever. Another alternative will later be put to a vote. This is why there is a connection between the two models: when an outcome far from the median voter outcome is proposed, a majority of players will prefer to reject the current proposal and wait for the arrival of a proposal closer to the median voter outcome.

The following proposition confirms that intuition and shows that when agents are patient, the solution gets close to the median voter outcome.

> **Proposition 7**: *Let preferences satisfy Assumption 3 with* v *concave.*[15] *Let* $m = int(n+1)/2$. *If* n *is odd, then* m *is the median voter and when* δ *tends to* 1, *in equilibrium, only proposals close to* θ_m *are accepted.*

To get some intuition for the result, assume that the set of accepted outcomes consists of an interval $A = [\underline{x}, \bar{x}]$.[16] Then A must contain θ_m. Indeed assume $\bar{x} < \theta_m$, then any individual $i \geq m$ would get utility at most $u_i(\bar{x})$, hence they would also accept any outcome in $[\bar{x}, \theta_m]$. Since these individuals form a majority, such offers are accepted, leading to a contradiction. It follows that A must contain θ_m. But, how large can A be? We will show that A must be small as δ tends to 1. Assume by contradiction that A is large. Then for very patient individuals, the expected value of voting against the proposed alternative is

$$E[u_i(x) \mid x \in A]$$

[15] *The assumption that* v *is concave is not necessary. It greatly simplifies the proof however.*

[16] This will easily follow from the concavity of v.

which, because preferences are single peaked, is strictly larger than $\min(u_i(\underline{x}), u_i(\bar{x}))$.[17] So no individual votes for both $\underline{x}$ and $\bar{x}$, and there cannot be a (strict) majority for both $\underline{x}$ and $\bar{x}$. A complete proof appears in Appendix.

5.2 Qualified majority rules and the moderate voters outcome.

We now turn to qualified majority rules, and restrict attention to the simpler case where v is concave. While standard voting models would not be predictive anymore (proposals get harder to defeat), our model remains predictive (when players are patient). The following Proposition provides a full characterization of the equilibrium outcome for the case where v is concave. It shows that, as for the unanimity case, the preferences of only two individuals matter. But as one gets from unanimity to the majority rule, the decisive individuals become more and more moderate.

> **Proposition 8**: *Let preferences satisfy Assumption 3 with v concave. Consider the qualified majority rule $n_0 > Int(n/2)$. When δ* tends to 1, *the equilibrium outcome coincides with that obtained when only players n_0 and $n - n_0 + 1$ are present.*

To get some intuition for the result, let $A = [\underline{x}, \bar{x}]$ denote the acceptance set when only players n_0 and $i_0 = n - n_0 + 1$ are present. When δ tends to 1, this set tends to the Nash bargaining solution between these two players, and we have $\theta_{i_0} < \underline{x} < \bar{x} < \theta_{n_0}$. Let us check what happens when the other players are added. As for the unanimity case, players $i \in \{n - n_0 + 1, .., n_0\}$ accept all proposals in A. Now observe that since v is concave, rejecting a proposal cannot yield more than $u_i(\widetilde{x})$ where $\widetilde{x} = E[x \mid x \in A]$. It follows that players with a bliss point lower than θ_{i_0} do not accept all proposals in

[17] The assumption that $\underline{x}$ and $\bar{x}$ are far apart from each other as δ goes to 1 guarantees that conclusion.

A, but accept all proposals in $[\underline{x}, \widetilde{x}]$. And similarly, all players with a bliss point higher than θ_{n_0} accept all proposals in $[\widetilde{x}, \bar{x}]$. Hence for any $x \in A$, there is a qualified majority of at least n_0 players that accepts x. The full proof is more complex, because we need to show that the result holds for any equilibrium. The proof is relegated to the Appendix.

5.3 When X is two-dimensional.

Voting theory has been successful in dealing with one-dimensional settings and single peaked preference. When one departs from these assumptions however, the existence of a Condorcet winner is not guaranteed, and voting theory loses its force. We wish to illustrate in this Section that our model can provide novel insights whether a Condorcet winner exists or not, and more generally, whether the core is empty or not.

Consider first settings in which the core is empty. Then, as illustrated by Proposition 3, we cannot expect that the set of accepted agreement would converge to a small set (because if it did, then other agreements would be accepted as well). Nevertheless, our model provides a set prediction: not all proposals get accepted, and one can characterize the set of proposals which are indeed accepted. We will provide in Example 1 below a simple illustration in a two dimensional setting.

Consider next settings in which the core is non-empty. Then considering patient players, our model may provide a way to select among the possible elements of the core: this is illustrated in Proposition 4 for example where the Nash bargaining outcome was selected under unanimous consent rule and rich proposal space. More generally, it should be clear that when δ tends to 1, if the set of accepted agreement gets small, then it must be an element of the core (if not, then players would prefer to wait for the occurrence of the proposal in the core that dominates the supposed solution). In addition, under the unanimity rule, it should also be clear that when δ tends to 1, the set of accepted agreements should be small in any equilibrium. So, if not too many equilibria arise (as in the context of Proposition 4) our model permits to select a subset of the core. Under other termination rules however, three is no guarantee that the limit set of accepted agreements (if not small) will be in the core. The reason is that an outcome may be accepted by a large majority of players because some players may fear that another outcome unfavorable to them may be accepted by (another) large majority of players. Example 2 below will illustrate this phenomenon, showing that the limit set of accepted agreements may be large, despite the fact that there exists a unique Condorcet winner.

Example 1: There are three players, $i = 1, 2, 3$. The preference of player i is defined by $u_i(x) = v - a \mid x - \theta_i \mid^k$ with $k > 1$, where $\mid . \mid$ is the euclidean distance. Bliss points θ_i are located on the corners of an equilateral triangle, and we let $X = Co(\theta_1, \theta_2, \theta_3)$ denote the surface defined by this triangle. Proposals are assumed to be distributed uniformly on X.

Under unanimous consent (and patient individuals), the unique symmetric equilibrium outcome lies close to the point x^* that is equidistant from all θ_i. Under simple majority rule, there is no Condorcet winner. By symmetry, player i cannot expect more than $u_i(x^*)$. (This is due to the concavity of $d \to v - ad^k$ for $k > 1$ and that by symmetry the expected location of accepted proposals is x^*.) All outcomes x such that $\mid x - \theta_i \mid \leq \mid x^* - \theta_i \mid$ for at least two individuals are accepted. The actual acceptance is larger.[18] More generally, solutions can be computed as a function of the locus of the bliss points. This example should be contrasted with that of Baron (1991) who uses Baron-Ferejohn (1989)'s bargaining model, and predicts that in equilibrium, the proposer gives just enough to ensure that one (and only one - this is enough to pass to proposal) other party accepts and obtains the

[18] When k is close to 1 this is the actual acceptance set. For $k > 1$, we conjecture that the larger k the larger the acceptance set.

rest of the surplus.[19]

Example 2: There are four individuals $i = 1, ..4$ whose bliss points $\theta_1, ...\theta_4$ are located on the corners of a square, $X = Co(\theta_1, \theta_2, \theta_3, \theta_4)$. Proposals are distributed uniformly on the surface X of the square.

Under unanimous consent, the unique symmetric equilibrium outcome lies close to the point x^* equidistant from all θ_i. Under simple majority rule (three individuals have to say yes), x^* is the only point in the core.[20] Yet, It can be checked that when k is large enough, the set of accepted proposals remains large, even as δ tends to 1. (TO BE COMPLETED)

6 Imperfect control over offers

Although we view the lack of control over offers as an essential ingredient of any bargaining process, agents may have *some* control over offers. How does our analysis extend to the case where players have some control over offers? For simplicity, we assume below that one and only one agent, say party 1, has some control over offers.

We will distinguish between two types of imperfection. In the first one, we assume that player 1 can perfectly control offers, but only infrequent so. In the second one, we assume that player 1 may only influence the distribution over offers, but this distribution remains characterized by a density with full support on X.

We make the following observations: under the first imperfection, player 1 may derive a substantial benefit from control, and when players are patient, this benefit is larger under unanimous consent than under qualified majority rules.

[19] In the case of a surplus of size one to be shared, transferable utilities and very patient players, accepted proposals are of the form $(2/3, 1/3, 0)$.

[20] Any point located outside the triangle formed the bliss points of three players can be destabilized by the coalition of these three players.

In contrast, under the second imperfection, as discounting tends to one, player 1's ability to influence the distribution over offers has vanishing value in the unanimous consent case, while it may have positive value under qualified majority rules.

Infrequent control.

We make two observations. First, we show that even if control is infrequent, if it is perfect, then player 1 may derive a substantial benefit from it. Second, we show that if control is imperfect, in the sense that (i) player 1 may affect the distribution over offers, but (ii) this distribution remains characterized by a density with full support on X, then, as discounting tends to one, player 1's ability to influence the distribution over offers has vanishing value.

To formalize the first imperfection, we assume the following:

> A1: At at any date with probability $1-p$, the offer is drawn from X according to f, and with probability p, player 1 has complete control over the offer made in X.

The following Proposition shows that under unanimity rule, when players are very patient, player 1's ability to control offers allows him to obtain all the surplus.

> **Proposition 9:** *Assume A1 holds, and consider the unanimity rule case. Let* $\underline{v}_1(\delta)$ be the smallest equilibrium value for player 1 when players discount future payoffs with discount factor δ. When δ approaches 1, $\underline{v}_1(\delta)$ *approaches the highest feasible value for player* 1.

Intuitively, when players are patient, the acceptance set is very small, that is, only proposals that fall in a small neighborhood of the equilibrium value (say v) are accepted. Besides, at dates where player 1 happens to have full control over offers, he can exploit this control by picking an offer that is accepted for sure. These two observations imply that conditional on being accepted, a proposal has most likely be generated by player 1. So the game has a structure similar to a bargaining game where only player 1 would be making offers, and as a result, player 1 is able to extract all surplus.

In contrast, if only a (qualified) majority is required, then it cannot be that player 1 extracts all surplus: if it were the case, then all players other than 1 would accept any offer (giving positive payoff), hence in the event player 1 does not have full control, agreement would be immediate, whatever proposal is drawn, hence each player $i \neq 1$ would obtain an expected payoff at least equal to $(1-p) \Pr X_0 E[u_i(x) \mid x \in X_0] > 0$,[21] leading to a contradiction. The exact effect of the control of one player in the (qualified) majority case remains to be determined.

We now turn to a formal proof of Proposition 9

Proof of Proposition 9: Fix δ and consider the equilibrium that gives lowest payoff to party 1, and denote by v this equilibrium value. For any $w = (w_1, ..., w_n)$, let $\bar{u}_1(w)$ denote the maximum payoff player 1 can obtain when each player $j \neq 1$ gets w_j. At any date where player 1 makes an offer, he can secure $\bar{u}_1(\delta v)$. It follows that

$$v_1 \geq p\bar{u}_1(\delta v) + (1-p)\delta v_1$$

which implies

$$\bar{u}_1(\delta v) - \delta v_1 \leq (1-\delta)v_1/p.$$

It follows that v is not further away from the frontier than a term comparable to $1-\delta$. So (i): even in events where player 1 does not control offers, player j cannot expect more $\delta v_j + k(1-\delta)$ for some $k > 0$, and (ii) the probability of acceptance P (conditional on player 1 not controlling offers) is at most comparable to $1-\delta$. Also, in events where player 1 makes proposal, party

[21] Remember that $X_0 = \{x \in X \mid u_i(x) > 0 \text{ for all } i\}$.

j cannot get more than δv_j. As a consequence, we have:

$$v_j \leq \delta v_j + (1-p)P[k(1-\delta)]$$

hence $v_j \leq (1-p)Pk$, which is close to 0 when δ is close to 1 because the probability of acceptance P is close to 0. **Q. E. D.**

Imperfect influence over offers.

To formalize the second imperfection, assume that at any date player 1 may take an action a, not observable to other agents, that affects the distribution over the proposals made in any period, and denote by $g(. \mid a)$ the distribution over proposals induced by action a.[22] The following Proposition shows that in the face of extremely patient players, the ability to influence offers has no value under the unanimity rule.

> **Proposition 10:** *Assume* $g(. \mid a)$ is smooth and bounded away from 0 for all a. Under the unanimity rule, *when δ tends to 1, equilibrium values tend to v^*.*

Proof: In a stationary equilibrium, the action chosen by player 1 is the same in every period. For any fixed a, and in particular for the action which player 1 finds optimal, the argument of Proposition 1 applies, showing that equilibrium values must tend to the Nash solution. **Q. E. D.**

The main insight behind this proposition is that if the set of accepted proposals is very small, then conditional on acceptance, the distribution

[22] For example, one possible assumption is that with probability $1-p$, the distribution over offers is, as before, described by a density $f(x)$. With probability p however, player 1 has the option to choose a pair $a = (\beta, z) \in [0, \bar{\beta}] \times X$, that induces a distribution over proposals $g(x \mid \beta, z) = g_0 e^{-\beta(x-z)^2} f(x)$, where g_0 is a normalizing constant. $z \in X$ should be interpreted as a target proposal, $\beta \in [0, \bar{\beta}]$ as the agent's effort to control offers, and $\bar{\beta}$ as the extent to which party 1 can control offers.

over proposals looks like a uniform distribution, whatever actions players undertake to influence the distribution. So these actions have no effect on the outcome as players get very patient.

Under qualified majority rules, the effect of control can be quite different however. We have seen that under the qualified majority rule, if the space of proposals is rich, then the set A of proposals that are accepted could be quite large. One interesting corollary of this observation is that even if players are very patient, having some influence over the proposals made can now be valuable.

This observation however does not apply to the case where the space of proposal is one dimensional, because then, even when only a simple majority is required, the set of proposals that are accepted remains small when players are patient.

Discussion

The propositions above have characterized limit outcomes as players get very patient. When players are not very patient, the set of accepted agreement becomes large, even under the unanimity rule. This implies that for not very patient players, having some influence over the distribution over proposals may be effective, and players having the strongest ability to influence proposals, or target specific utility levels for their opponent should be able to extract more surplus. What Proposition 10 shows is that patience and veto power from other parties limit the effect of control.

Control or influence over proposals can thus be viewed as a source of bargaining power, that complements the traditional view that (in complete information models) bargaining power is driven by relative patience (Rubinstein) and by the relative frequency with which parties make offers (as in the random proposer model of Binmore).

One interesting corollary of the above observations (and a possible avenue for further research) has to do with the various ways parties might try to influence proposals. One can think of lobbying efforts, or in large groups, as efforts to form coalitions which depending on their size will carry more weight in influencing proposals in one way or another: forming a coalition may help its members controlling the process over offer generation. Our model suggests that such efforts will be more valuable (hence more likely to occur) when agents are not too patient and other players do not have veto power.

7 Appendix

Proof of Proposition 7: Let $\underline{x}$ (respectively $\bar{x}$) denote the lowest proposal accepted in equilibrium. Let $d = \bar{x} - \underline{x}$, and recall that A is the set of proposals accepted in equilibrium. We have shown in the main text that we must have $\underline{x} \leq \theta_m \leq \bar{x}$. We will show that when δ tends to 1, d must shrink to 0.

We first show that A must be an interval. Let $\widetilde{x} = E[x \mid x \in A]$. Since v is concave, $\delta v_i \leq \lambda u_i(\widetilde{x})$, hence all parties (unanimously) accept $\widetilde{x}$. Since v is single-peaked, parties that accept $\underline{x}$ and $\widetilde{x}$ also accept any proposal in $[\underline{x}, \widetilde{x}]$. So there must be a majority voting for these proposals. A similar argument applies to proposals in $[\widetilde{x}, \bar{x}]$. So $A = [\underline{x}, \bar{x}]$.

We now show that $\lambda \nearrow 1$ as δ tends to 1. Assume λ does not tend to 1. Then there exists a sequence of discounts $\delta_k \nearrow 1$ and equilibria such that $\lambda_k < \lambda < 1$. Since $\delta v_i \leq \lambda u_i(\widetilde{x})$, hence all parties (unanimously) accept all proposals in a neighborhood of $\widetilde{x}$ (of size comparable to $1-\lambda$). It follows that $\Pr A$ does not tend to 0, contradicting the premise that λ remains bounded away from 1.

Now assume that d does not tend to 0. Since v is concave, and since the distribution over proposals has full support, there must exists $\mu > 1$ such that

$$E[u_i(x) \mid x \in A] > \mu \min(u_i(\bar{x}), u_i(\underline{x})]. \tag{9}$$

Since, at least one party, say i_0, must accept simultaneously $\underline{x}$ and $\bar{x}$, we must also have

$$\min(u_{i_0}(\bar{x}), u_{i_0}(\underline{x}) \geq \lambda E[u_{i_0}(x) \mid x \in A],$$

which, combined with inequality (9) implies that $\lambda < 1/\mu$, contradicting the fact that λ must tend to 1. So d must tend to 0.

Proof of Proposition 8: Let v_i denote player i's equilibrium value, and let A denote the equilibrium acceptance set. Also let $\bar{x}$ (respectively $\underline{x}$) denote the highest (respectively lowest) proposal accepted in equilibrium. Following the steps of Proposition 7, we have:

$$\delta v_i = \lambda E[u_i(x) \mid x \in A]$$

where $\lambda = \frac{\delta \Pr A}{1-\delta+\delta \Pr A} < 1$.

Define $g(\theta, x)$ as in Proposition 7 and let $i_0 = n - n_0 + 1$.

(i) We check that $g(\theta_{i_0}, \bar{x}) = 0$.

Indeed, if $g(\theta_{i_0}, \bar{x}) > 0$, then for any $\theta \geq \theta_{i_0}$, $g(\theta, \bar{x}) > 0$.[23] Hence by continuity, there are proposals $x > \bar{x}$ that are accepted by at least n_0 individuals, contradicting the premise that $\bar{x}$ is the largest proposal accepted in equilibrium. So $g(\theta_{i_0}, \bar{x}) \leq 0$. Now if $g(\theta_{i_0}, \bar{x}) < 0$, then $g(\theta, \bar{x}) < 0$ for all $\theta < \theta_{i_0}$, hence $\bar{x}$ would not be accepted by the qualified majority n_0.

(ii) Similarly, it is easy to check that $g(\theta_{n_0}, \underline{x}) = 0$.

(iii) We now check that $A = [\underline{x}, \bar{x}]$. Let $\widetilde{x} = E[x \mid x \in A]$. Since v is concave, $\delta v_i \leq \lambda u_i(\widetilde{x})$, hence proposal $\widetilde{x}$ is accepted unanimously. Any party who accepts both $\underline{x}$ and $\widetilde{x}$ must also accept all proposals in $[\underline{x}, \widetilde{x}]$. So, since there is a qualified majority for $\underline{x}$, there is also a qualified majority for any proposal $x \in [\underline{x}, \widetilde{x}]$. A similar argument applies to proposals in $[\widetilde{x}, \bar{x}]$.

(iv) It thus follows that $A = [\underline{x}, \bar{x}]$, and that $\bar{x}$ and $\underline{x}$ are fully characterized by $g(\theta_{n_0}, \underline{x}) = 0$, and $g(\theta_{i_0}, \bar{x}) = 0$. The solution is thus identical to that which would obtain if only i_0 and n_0 were present. Also note that, by

[23] This is because for any $\theta < \bar{x}$, $\frac{\partial g}{\partial \theta}(\theta, \bar{x}) > 0$, and because for any $\theta \geq \bar{x}$, $g(\theta, \bar{x}) > 0$.

an argument similar to that of Proposition 7, when δ tends to 1, $\bar{x} - \underline{x}$ must tend to 0.

References

[1] Banks, J. and Duggan, J. (2000) "A Bargaining Model of Collective Choice", *American Political Science Review*, **94**, 73-88.

[2] Baron, D. (1991) "A Spatial Bargaining Theory of Government Formation in Parliamentary Systems" *American Political Science Review*, **85** 137-164.

[3] Baron, D. (1996) "A Dynamic Theory of Collective Good Programs", *American Political Science Review*, **83**, 1181-1206.

[4] Baron, D. and Ferejohn (1989), "Bargaining in Legislatures", *American Political Science Review*, **90**, 316-330.

[5] Binmore, K. (1987) "Perfect Equilibria in Bargaining Models", in *The Economics of Bargaining*, ed. K. Binmore and P. Dasgupta, Oxford: Basil Blackwell, 77-105.

[6] Binmore, K., A. Rubinstein and A. Wolinsky (1986): "The Nash Bargaining Solution", *RAND Journal of Economics*, **17**, 176-188.

[7] Cai, H. (2000) "Delay in Multilateral Bargaining under Complete Information", *Journal of Economic Theory*, **93**, 260-276.

[8] Cai, H. (2002) "Uncertainty of Commitment in Multilateral Bargaining", mimeo UCLA

[9] Nash, J. (1950), "The bargaining problem", *Econometrica* **18**, 155-162.

[10] Rubinstein, A. (1982), "Perfect Equilibrium in a Bargaining Model", *Econometrica* **50**, 97-109.