Calibrated Clustering and Analogy-Based Expectation Equilibrium *

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Abstract

Families of normal-form two-player games are categorized by players into K analogy classes applying the K-means clustering technique to the data generated by the distributions of opponent's behavior. This results in Calibrated Analogy-Based Expectation Equilibria in which strategies are analogy-based expectation equilibria given the analogy partitions and analogy partitions are derived from the strategies by the K-means clustering algorithm. We discuss various concepts formalizing this, and observe that distributions over analogy partitions are sometimes required to guarantee existence. Applications to games with linear best-responses are discussed highlighting the differences between strategic complements and strategic substitutes.

Keywords: K-mean clustering, Analogy-based Expectation Equilibrium

JEL Classification Numbers: D44, D82, D90

1 Introduction

Many economists recognize that the rational expectation hypothesis that is central in solution concepts such as the Nash equilibrium seems very demanding, especially in complex multi-agent environments involving lots of different situations

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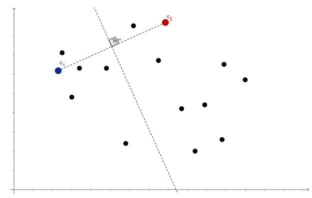
(games, states or nodes, depending on the application). Several approaches have been proposed to relax it. When the concern with the hypothesis is that there are too many situations for players to fine tune a specific expectation for each such situation, a natural approach consists in allowing players to lump together situations into just a few categories, and only require that players form expectations about the aggregate play in each category (as opposed to forming a different expectation for each situation separately).

The analogy-based expectation equilibrium (Jehiel, 2005) is a solution concept that has been proposed to deal with this. In addition to the usual primitives describing a game form, players are also endowed with analogy partitions, which are player-specific ways of partitioning situations or contingencies in the grand game. In equilibrium, the expectations in each analogy class correctly represent the aggregate behavior in the class, and players best-respond as if the behavior in every element of an analogy class matched the expectation about the aggregate play in the corresponding analogy class. This approach has been developed and applied to a variety of settings (see Jehiel (2022) for a recent account of this), but in almost all these developments, the analogy partitions are taken as exogenous.

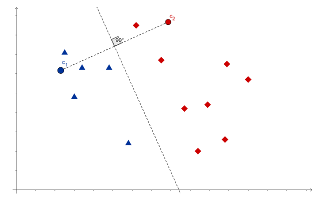
In this paper, we propose endogenizing the choice of analogy partitions made by the players on the basis of simple clustering techniques routinely used in Machine Learning. Specifically, we will rely on the K-means clustering technique used in Machine Learning to cluster datapoints into a pre-specified number K of categories. The K-means technique was originally proposed by Steinhaus (1957), Lloyd (1957) and MacQueen (1967).[1] Roughly, it works as follows. Datapoints are the primitives, and the clustering problem consists in partitioning the datapoints into K clusters with representative points for each cluster defined so that the original datapoints are best approximated by the representative points in their cluster. In general, the retained criterion is that of minimizing the sum of the prediction errors, where errors are measured using some notion of distance (or divergence) to represent how far a datapoint is from the representative point in the cluster. The most widely used criterion is the sum of the squared Euclidean distance of the datapoints to the representative points, in which case the criterion

[1] See Jain et al. (1999) for a comprehensive survey on data clustering or Blömer et al. (2016) for a theoretically oriented survey on K-means clustering.

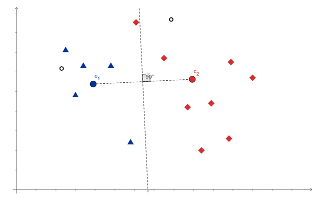
amounts to minimizing the total variance. K-means clustering has also been extended to Bregman divergences (that include the Kullback-Leibler divergence and the squared Euclidean distance) by Banerjee et al. (2005). Solving the clustering problem is hard (NP-complete) and practitioners most of the time rely on the following simple algorithm to approximate its solution. Start with K initial centroids, assign each data point to the closest one. Based on this assignment, new centroids are formed (defined as the mean points or Barycenters of the data points assigned to the various centroids). Repeat the process until an iteration is reached and the centroids do not change anymore. This algorithm is simple to implement and converges very fast in practice. It always converges to a partitioning which is a local solution to the clustering problem. It is illustrated in the following Figure.[2]



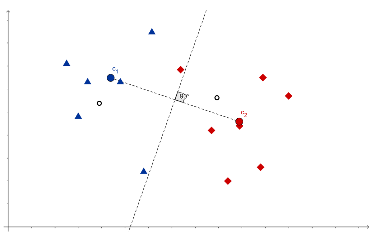
(a) Iteration 0. Random Initialization: c_1, c_2



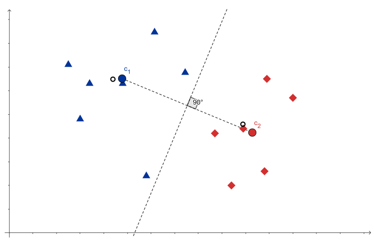
(b) Iteration 1. Step 1. Assigning data to closest c



(c) Iteration 1. Step 2. Recompute centroids c_1 and c_2



(d) Iteration 2. Step 1 and 2.



(e) Iteration 3. Step 1 and 2. Stop.

Figure: A simple illustration of the K-means algorithm at work

In this paper, we consider a strategic environment consisting of different normal form two-player games drawn by nature according to some prior distribution where we have in mind that the various games are played at many different times by many different subjects. In each of the normal form games $\omega \in \Omega$, player $j = 1, 2$ has the same action set A_j. An analogy partition for player i takes the form of a partition

[2]In the figure the filled dots represent the new centroids, while the empty dots are the old ones. Different colors (or shapes) represent different clusters formed at step 1 of an iteration, while the dashed line separates the points closest to each centroid after these have been recomputed in step 2.

of the set of games Ω, which is used by player i to assess the behavior of player j in the various games. From the clustering perspective, the data points accessible by players consist of the empirical frequencies of past play of the subjects assigned to the role of the opponent in the various games. That is, a typical data point for player i consists of an element of ΔA_j for each of the games ω. To make sense of these data points (and prior to knowing which specific game ω will apply), a subject assigned to the role of player i is viewed as clustering these data points into an exogenously given (typically small) number of categories. When called (later) to pick an action in a randomly selected game ω, this subject then identifies the behavior of his opponent in this game ω with the representative expectation that comes out from the clustering stage, and best-responds to it. This in turn generates new data points, and we are interested in the steady states - referred to as calibrated analogy-based expectation equilibria - generated by such dynamic processes.

Intuitively, the calibrated analogy-based expectation equilibria (C-ABEE) can be described as profiles of analogy partitions and strategies such that i) given the analogy partitions, players' strategies form an analogy-based expectation equilibrium and ii) given the strategies, clustering leads players to adopt the analogy partitions considered in steady state.

Different formalizations of clustering can be considered whether we insist on an exact resolution of the clustering problem (variance minimization, say) or whether we consider a possible outcome of the K-means algorithm leading only to a local optimality condition. But, no matter what approach to clustering is adopted, a key observation is that it may not be possible in some cases to have a steady state with a single analogy partition for each player. This is so because unlike in the usual clustering problem, there is here an extra endogeneity of the dataset. A change in analogy partitions may affect the adopted strategies through the working of the analogy-based expectation equilibrium, which in turn may affect how clustering is done. This extra channel from the clustering to the dataset makes it sometimes impossible to have a calibrated analogy-based expectation equilibrium with a single analogy partition for each player. This will be illustrated with a simple example involving three normal form games and binary action spaces.

The above observation leads us to extend the basic definition of C-ABEE to allow for distributions of analogy partitions in which each analogy partition in the support is required to solve (either locally or globally) the clustering problem for that player and the strategies now also parameterized by the chosen analogy partition satisfy the requirements of the analogy-based expectation equilibrium appropriately extended to cope with distributions of analogy partitions. We refer to such an extension as a calibrated distributional analogy-based expectation equilibrium (CD-ABEE).

We show that in finite environments (i.e. environments such that there are finitely many normal form games and finitely many actions for each player), there always exists at least one CD-ABEE. We also provide a learning foundation to CD-ABEE, establishing that they are the steady states of learning dynamics involving populations of players randomly matched in each period where the payoffs as well as the data used at the clustering stage are subject to player-specific perturbations.

In the last part of the paper, we consider an application to families of games with linear best-responses parameterized by the magnitude of the impact of opponent's action on the best-response (a one-dimensional parameter). We analyze separately the case of strategic complements and the case of strategic substitutes allowing us to cover applications such as Bertrand or Cournot duopoly with product differentiation, linear demand and constant marginal costs, or moral hazard in teams. In this part, we consider a continuum of games and a continuum of actions, which simplifies the exposition of the results.

Our main results in the application part are as follows. In the case of strategic complements, one can always find a Calibrated ABEE in which a single analogy (interval) partition is used by the players. Our proof is constructive, and we exhibit one such C-ABEE before providing a characterization of all of them. More precisely, we show this when using the local optimality version of clustering whereas for the global optimality one, we are able to provide a similar result, only when the interaction effect (the strategic part in the best-response) is not too strong. By contrast, in the case of strategic substitutes, we show that there is no Calibrated ABEE in which a single analogy interval partition is used by the players.

When heterogeneous analogy partitions are required in CD-ABEE, we note that faced with the same objective datasets and the same objective constraints (as measured by the number of classes), players must be processing information in a heterogeneous way in equilibrium. Our derivation of this insight follows from the strategic nature of the interaction (leading, as highlighted above, the dataset to be affected by how players categorize games). It should be contrasted with other possible motives of heterogeneity, for example, based on the complexity of processing rich datasets.[3]

It should be noted that throughout the paper, we take as exogenous the number of categories a player considers for clustering purposes. From a Machine Learning perspective, a natural next step would be to endogenize this number. In this regard, while various approaches have been proposed, it should be mentioned that there is no consensus on how to endogenize this number, and it is still a subject of active research in Machine Learning. From another more psychological perspective, it has long been recognized that there are severe constraints on how many items human beings can remember in short-term memory (see in particular Miller (1956) for pioneering research on this). As the number of categories in our setting can naturally be related to the number of items remembered from the dataset, this psychological perspective would lead to treat this number as exogenous, as we do in this paper.

In the rest of the paper, we develop the framework (solution concepts, existence results, learning foundation) in Section 2. We discuss the application to games with linear best-responses in Section 3. We conclude in Section 4.

1.1 Related Literature

This paper belongs to a growing literature in behavioral game theory, proposing new forms of equilibrium to capture various aspects of misperceptions or cognitive limitations. While some papers in this strand posit some misperceptions of the players and propose a corresponding notion of equilibrium (see Eyster-Rabin (2005) on misperceptions about how private information affects behavior, Spiegler (2016) on misperceptions on the causality links between variables of interest or Esponda-Pouzo (2016) for a more abstract and general formulation of misspecifi-

[3] Such forms of heterogeneity are implicitly suggested in Aragones et al (2005) (when they highlight that finding regularities in complex datasets is NP-hard) or Sims (2003) (who develops a rational inattention perspective to model agents who would be exposed to complex environments).

cations), other papers motivate their equilibrium approach by the difficulty players may face when trying to understand or learn how their environment behaves (see Jehiel (1995) on limited horizon forecasts, Osborne-Rubinstein (1998) on sampling equilibrium, Jehiel (2005) on analogical reasoning or Jehiel-Samet (2007) on coarse reinforcement learning). Our paper has a motivation more in line with the latter, but it adds structure on the coarsening of the learning based on insights or techniques borrowed from machine learning (which the previous literature just mentioned did not consider).

This paper also relates to papers dealing with coarse or categorical thinking in decision-making settings (see, in particular, Fryer and Jackson (2008) for such a model used to analyze stereotypes or discrimination, Peski (2011) for establishing the optimality of categorical reasoning in symmetric settings or Al-Najjar and Pai (2014) and Mohlin (2014) for models establishing the superiority of using not too fine categories in an attempt to mitigate overfitting or balance the bias-variance trade-off). While our paper considers a clustering technique (K-means) not discussed in those papers, another essential difference is that in our setting the data-generating process is itself affected by the categorization due to the strategic character of our environment.[4]

2 Theoretical setup

2.1 Strategic environment

We consider a finite number of normal form games indexed by $\omega \in \Omega$ where game ω is chosen (by Nature) with probability $p(\omega)$. To simplify the exposition, we restrict attention to games with two players $i = 1, 2$, and we refer to player j as the player other than i.[5] In every game ω, the action space of player i is the same and denoted by A_i. It is assumed in this part to be finite. The payoff of player i in game ω is described by a von Neumann-Morgenstern utility where $u_i(a_i, a_j, \omega)$

[4]Some papers consider categorization in games (see in particular Samuelson (2001) or Mengel (2012)) with the view that the strategy should be measurable with respect to the categorization. This is somewhat different from the expectation perspective adopted here.

[5]The framework, solution concept and analysis extend in a straightforward way to the case of more than two players, provided the behavioral data of the various players are treated separately from one another at the clustering stage. If bundling occurs also across players (as permitted in Jehiel (2005)), additional work is required.

denotes the payoff obtained by player i in game ω if player i chooses $a_i \in A_i$ and player j chooses $a_j \in A_j$. Let $p_i \in \Delta A_i$ denote a probability distribution over A_i for $i = 1, 2$. With some abuse of notation, we let:

$$u_i(p_i, p_j, \omega) = \sum_{a_i, a_j} p_i(a_i) p_j(a_j) u_i(a_i, a_j, \omega)$$

denote the expected utility obtained by player i in game ω when players i and j play according to p_i and p_j, respectively.

We assume that players observe the game ω they are in. A strategy for player i is denoted $\sigma_i = (\sigma_i(\omega))_{\omega \in \Omega}$ where $\sigma_i(\omega) \in \Delta A_i$ denotes the (possibly mixed) strategy employed by player i in game ω. The set of player i's strategies is denoted Σ_i, and we let $\Sigma = \Sigma_i \times \Sigma_j$.

A Nash equilibrium is a strategy profile $\sigma = (\sigma_i, \sigma_j) \in \Sigma$ such that for every player i, $\omega \in \Omega$, and $p_i \in \Delta A_i$,

$$u_i(\sigma_i(\omega), \sigma_j(\omega), \omega) \geq u_i(p_i, \sigma_j(\omega), \omega).$$

2.2 Analogy-based expectation equilibrium

Players are not viewed as being able to know or learn the strategy of their opponent for each game ω separately as implicitly required in Nash equilibrium. Maybe because there are too many games ω, they are assumed to learn the strategy of their opponent only in aggregate over collections of games, referred to as analogy classes. Throughout the paper, we impose that player i considers K_i different analogy classes, where K_i is kept fixed. We have in mind that K_i is no greater (and typically smaller) than $|\Omega|$, the number of possible normal form games. We refer to $\mathcal{K}_i$ as the set of partitions of Ω with K_i elements. Formally, considering for now the case of a single analogy partition for each player i (this will be extended to distributional approaches later on), we let $An_i = \left\{\alpha_i^1, \dots \alpha_i^{K_i}\right\}$ denote the analogy partition of player i. It is a partition of the set Ω of games with K_i classes, hence an element of $\mathcal{K}_i$. For each $\omega \in \Omega$, we let $\alpha_i(\omega)$ denote the (unique) analogy class to which ω belongs.

$\beta_i(\alpha_i) \in \Delta A_j$ will refer to the (analogy-based) expectation of player i in the analogy class α_i. It represents the aggregate behavior of player j across the various games ω in α_i.

We say that β_i is *consistent* with σ_j whenever for all $\alpha_i \in An_i$,

$$\beta_i(\alpha_i) = \sum_{\omega \in \alpha_i} p(\omega)\sigma_j(\omega) / \sum_{\omega \in \alpha_i} p(\omega).$$

In other words, consistency means that the analogy-based expectations correctly represent the aggregate behaviors in each analogy class when the play is governed by σ.

We say that σ_i is a *best-response* to β_i whenever for all $\omega \in \Omega$ and all $p_i \in \Delta A_i$,

$$u_i(\sigma_i(\omega), \beta_i(\alpha_i(\omega)), \omega) \geq u_i(p_i, \beta_i(\alpha_i(\omega)), \omega).$$

In other words, player i best-responds in ω as if player j played according to $\beta_i(\alpha_i(\omega))$ in this game.

Definition 1. *Given the strategic environment and the profile of analogy partitions $An = (An_i, An_j)$, σ is an analogy-based expectation equilibrium (ABEE) if and only if there exists a profile of analogy-based expectations $\beta = (\beta_i, \beta_j)$ such that for each player i (i) σ_i is a best-response to β_i and (ii) β_i is consistent with σ_j.*

This concept has been introduced with greater generality in Jehiel (2005) (allowing for multiple stages and more than two players) and in Jehiel and Koessler (2008) (allowing for private information).[6] Roughly, the premise is that players can only base their choice of strategy on the aggregate behaviors of their opponent in their various analogy classes. The proposed notion of best-response views the players as adopting the simplest representation of their opponent's strategy that is compatible with such aggregate statistics. Moreover, the consistency of the analogy-based expectations is viewed as the outcome of a learning process in which players would only focus on the aggregate behaviors of their opponent in each analogy class. An analogy-based expectation equilibrium can be thought of as a steady state of such a learning environment (see Jehiel (2022) for further discussions of the concept and subsection 2.6 for further elaborations on the learning dynamics we have in mind in the present context).

[6] See Jehiel (2022) for a definition in a setting covering both aspects and allowing for distributions over analogy partitions.

2.3 Calibrated clustering

The general idea behind clustering as considered in machine learning is to group (data) points by proximity into clusters. The K-means clustering algorithm that is very commonly used considers the square of the Euclidean distance as the notion of proximity, but other notions of proximity can be considered as well. In our problem, the objects to be clustered by player i concern the distributions of opponent j's actions over the different games. That is, for player i, the clustering (into K_i clusters) concerns $\sigma_j(\omega) \in \Delta A_j$ for the various games ω in Ω. We will be considering several notions of proximity that can be captured by a divergence function $d(p_j, p'_j)$ where $d(p_j, p'_j) > d(p_j, p''_j)$ indicates that p_j is less well approximated by p'_j than by p''_j.

Throughout the paper, we will consider for d either the square of the Euclidean distance $d(p_j, p'_j) = \left\|p_j - p'_j\right\|^2$ defined over $(\Delta A_j)^2$ as in the standard K-means approach, or the Kullback-Leibler divergence applied to distributions $d(p_j, p'_j) = \sum_{a_j} p_j(a_j) \ln \frac{p_j(a_j)}{p'_j(a_j)}$ which can be given a likelihood interpretation (see more on this below). In the application with linear best-responses to be developed later, the actions will take values in the set of real numbers, and we will be considering for d the square of the Euclidean distance applied to the mean of each distribution, i.e. $d(p_j, p'_j) = \left(E(p_j) - E(p'_j)\right)^2$.

A fundamental property of the (divergence) functions d just mentioned is:[7]

Lemma 1. *For $i = 1, 2$ and any subset α_i of Ω, let d be either the square of the Euclidean distance or the Kullback-Leibler divergence. Then*

$$\sum_{\omega \in \alpha_i} p(\omega \mid \alpha_i)\sigma_j(\omega) = \arg\min_{q \in \Delta A_j} \sum_{\omega \in \alpha_i} p(\omega \mid \alpha_i) d(\sigma_j(\omega), q).$$

This lemma implies that the best representative q of a cluster α_i should be the mean value of the points in the cluster,[8] for the purpose of minimizing the expected value of $d(\sigma_j(\omega), q)$ within the cluster α_i.[9] Interestingly, $\sum_{\omega \in \alpha_i} p(\omega \mid$

[7]This result is not new. For completeness, we have included a proof of it in the online appendix.

[8]Since the mass of data corresponding to $\sigma_j(\omega)$ would be proportional to $p(\omega)$, the mean has to respect the weighting shown in the lemma.

[9]Banerjee, Guo and Wang (2005) show that a necessary and sufficient condition for this to

$\alpha_i)\sigma_j(\omega)$ coincides with $\beta_i(\alpha_i) = \sum_{\omega\in\alpha_i} p(\omega)\sigma_j(\omega)/\sum_{\omega\in\alpha_i} p(\omega)$ as introduced above in the context of ABEE, which we will be using when interpreting our proposed solution concept later on (see the learning foundation section). But, the property derived in the lemma is also essential to prove the convergence of the K-means clustering algorithm in which K_i representatives are initially randomly drawn, and at each subsequent iteration of the algorithm, first points are allocated to the cluster with closest representative, then, a representative, identified with the mean, is determined in each cluster (see Introduction).[10] While such an algorithm always converges to what we refer to as a locally calibrated clustering, it may sometimes fail to solve fully the clustering problem defined as $\arg\min_{q_1,\dots q_{K_i}} \sum_{\omega\in\Omega} p(\omega) \min_{q\in\{q_1,\dots q_{K_i}\}} d(\sigma_j(\omega), q)$, where $q_1,\dots q_{K_i}$ are the representative points in the various clusters.

This discussion leads us to provide the following definitions, where for completeness we include in the online appendix (Lemma 3) a proof that global calibration implies local calibration.

Definition 2. *A partition An_i of Ω is locally calibrated with respect to σ_j iff for every classes α_i, α_i' of An_i and every $\omega \in \alpha_i$,*

$$d(\sigma_j(\omega), \beta_i(\alpha_i)) \leq d(\sigma_j(\omega), \beta_i(\alpha_i')).$$

It is globally calibrated with respect to σ_j iff

$$An_i \in \arg\min_{P_i\in\mathcal{K}_i} \sum_{c_i\in P_i} p(c_i) \sum_{\omega\in c_i} p(\omega \mid c_i) d(\sigma_j(\omega), \beta_i(c_i))$$

where, for all $c \subseteq \Omega$, $\beta_i(c) = \sum_{\omega\in c} p(\omega \mid c)\sigma_j(\omega)$.

We note that given σ_j, there always exists a partition that is globally calibrated with respect to σ_j. This is because in our finite environment there are finitely many partitions $P_i \in \mathcal{K}_i$ and at least one of them must minimize $\sum_{c_i\in P_i} p(c_i) \sum_{\omega\in c_i} p(\omega \mid$

hold is that d is a Bregman divergence.

[10]The function $\sum_{\omega\in\Omega} p(\omega) \min_{q\in\{q_1,\dots q_{K_i}\}} d(\sigma_j(\omega), q)$ where $q_1,\dots q_{K_i}$ are the representative points in the various clusters can be shown to monotonically decrease along the various steps of the algorithm, which can be used to prove the convergence of the algorithm (since this function constitutes a Lyapounov function for the dynamic process defined by the algorithm).

$c_i)d(\sigma_j(\omega), \beta_i(c_i))$. Since global calibration implies local calibration, this also shows the existence of a locally calibrated partition for any given σ_j.

From the viewpoint of the analogy-based expectation equilibrium, the local calibration of the analogy-based expectation An_i of player i means that player j's behavior in game $\omega \in \alpha_i$, i.e. $\sigma_j(\omega)$, is no less well approximated by $\beta(\alpha_i)$ than by any alternative $\beta_i(\alpha'_i)$ with $\alpha'_i \in An_i$. When d is the Kullback-Leibler divergence, it amounts to requiring that the likelihood of observing behaviors governed by $\sigma_j(\omega)$ is no smaller if the assumed behavior is $\beta_i(\alpha_i)$ than if it is any alternative theory $\beta(\alpha'_i)$, $\alpha'_i \in An_i$. It can be viewed as a stability idea in which, assuming the possible theories are $\beta_i(\alpha'_i)$ for the various $\alpha'_i \in An_i$, player i would find no reason to reassign any game ω to an analogy class other than the one it is assigned to in An_i.

Global calibration on the other hand means that in the face of data points as given by $\sigma_j(\omega)$ for $\omega \in \Omega$, the clustering into An_i is best for the purpose of solving

$$\arg \min_{q_1,\ldots q_{K_i}} \sum_{\omega \in \Omega} p(\omega) \min_{q \in \left\{q_1,\ldots q_{K_i}\right\}} d(\sigma_j(\omega), q). \tag{1}$$

Solving the full clustering problem as represented in (1) is known to be NP-hard in Computer Science, thereby leading the K-means clustering algorithms to be widely used in practice to find out the clusters. Given that such algorithms only ensure the finding of local optima, we believe that, with this perspective, it is meaningful to consider local calibration (even if for theoretical purposes global calibration would look like the more natural criterion).

2.4 Calibrated analogy-based expectation equilibrium

Combining the above definitions yields:

Definition 3. *A pair (σ, An) of strategy profile $\sigma = (\sigma_i, \sigma_j)$ and analogy partition profile $An = (An_i, An_j) \in \mathcal{K}_i \times \mathcal{K}_j$ is a locally (resp. globally) calibrated analogy-based expectation equilibrium iff (i) σ is an analogy-based expectation equilibrium given An and (ii) for each player i, An_i is locally (resp. globally) calibrated with respect to σ_j.*

The more interesting and novel aspect in this definition is the fixed point element linking analogy partitions to strategies and vice versa. With respect to the previous papers (using the ABEE framework), it suggests a way to endogenize the analogy partitions (given the numbers K_i and K_j of allowed analogy classes). With respect to the clustering literature, the novel aspect is that the set of points to be clustered ($(\sigma_j(\omega))_{\omega\in\Omega}$ for player i) is itself possibly influenced by the shape of the clustering, as captured by the analogy-based expectation equilibrium.

When either player 1 or 2 has a dominant strategy in all games $\omega \in \Omega$, there always exists a (locally or globally) calibrated ABEE. To see this, suppose player i has a dominant strategy in all ω. The behavior of player i coincides with the dominant strategy irrespective of the profile of analogy partitions. This ensures that on player j's side, the analogy partition can simply be obtained by using the standard clustering techniques applied to the exogenous dataset given by player i's dominant strategy in the various games. Once such a clustering is derived, the rest of the construction of a calibrated ABEE is easily derived.

When no player has a dominant strategy across all games $\omega \in \Omega$, we will now illustrate that there may be no (σ, An) that is a (locally or globally) calibrated analogy-based expectation equilibrium. The basic existence problem can be understood as follows. Starting from a profile of analogy partitions An, a strategy profile σ that is an analogy-based expectation equilibrium for An always exists (see the existence result in Jehiel (2005) or Jehiel and Koessler (2008)). On the other hand, as already mentioned, starting from a strategy profile σ, it is always possible to find profile(s) of analogy partitions that are calibrated with respect to σ. But, when the ABEE strategy profile σ varies with the analogy partitions, there is no reason why the induced compound correspondence would have a fixed point, thereby making the existence of a calibrated analogy-based expectation equilibrium as just defined sometimes impossible. We will address this existence issue by proposing a distributional approach (that parallels the introduction of mixed strategies in the context of Nash equilibria), but for now let us illustrate how calibrated analogy-based expectation equilibria may fail to exist.

Example 1. The following three games are played, each with probability $\frac{1}{3}$. The corresponding payoff matrices are given by:

$$\begin{array}{ccc} \omega_1 & L & R \\ U & (.,1) & (.,0) \\ D & (.,1) & (.,0) \end{array} \qquad \begin{array}{ccc} \omega_2 & L & R \\ U & (1+x,0) & (0,1) \\ D & (0,1) & (1,0) \end{array} \qquad \begin{array}{ccc} \omega_3 & L & R \\ U & (.,0) & (.,1) \\ D & (.,0) & (.,1) \end{array}$$

where $0 < x$ and the "." in the ω_1 and ω_2 games could take any value.

Proposition 1. *Assume that $K_1 = 2$ and $K_2 = 3$, and d is the square of the Euclidean distance. There is no locally calibrated ABEE when $x < 1$. There is a locally but no globally calibrated ABEE when $1 \leq x < 2$.*

Proof. 1) We first rule out the case in which $\alpha = \{\omega_1, \omega_3\}$ is the non-singleton analogy class for player 1. If so, $\beta_1(\alpha) = \frac{1}{2}L \oplus \frac{1}{2}R$ and in the only (Nash) equilibrium of ω_2, the Column player would play $\sigma_2(\omega_2) = \frac{1}{2+x}L \oplus \frac{1+x}{2+x}R$ so as to make the Row player indifferent between U and D. But, given that $d(R, \frac{1}{2+x}L \oplus \frac{1+x}{2+x}R) < d(R, \frac{1}{2}L \oplus \frac{1}{2}R)$ when $x > 0$, we would have $d(\sigma_2(\omega_3), \beta_1(\alpha)) > d(\sigma_2(\omega_3), \beta_1(\{\omega_2\}))$ invalidating the local calibration condition for game ω_3.

2) Consider next the case in which $\alpha = \{\omega_1, \omega_2\}$ is the non-singleton analogy class. In the corresponding ABEE, we should have that $\beta_1(\alpha)$ attaches probability at least $\frac{1}{2}$ to L (given that L is played in ω_1 and both ω_1 and ω_2 are equally likely), and thus the Row player should choose U in ω_2 implying that the Column player chooses R in game ω_2 (remember that $K_2 = 3$ implies that the Column player plays optimally in each game). But, then $d(\sigma_2(\omega_2), \beta_1(\omega_3)) = d(R, R) = 0 < d(R, \frac{1}{2}L \oplus \frac{1}{2}R) = d(\sigma_2(\omega_2), \beta_1(\alpha))$ (where $\beta_1(\alpha) = \frac{1}{2}L \oplus \frac{1}{2}R$ is derived from consistency), thereby invalidating the local calibration condition for game ω_2, which would have to be re-assigned to the analogy class $\alpha' = \{\omega_3\}$ instead of $\alpha = \{\omega_1, \omega_2\}$.

3) Consider last $\alpha = \{\omega_2, \omega_3\}$ as the non-singleton analogy class. In the corresponding ABEE, the strategy of the Column player in game ω_2 cannot be pure.[11] This implies that $\sigma_1(\omega_2) = \frac{1}{2}U \oplus \frac{1}{2}D$ so as to make the Column player indifferent between L and R. Now, for the Row player to be indifferent between U and D

[11] To see this, assume first by contradiction that $\sigma_2(\omega_2) = L$, then $\beta_1(\alpha) = \frac{1}{2}L \oplus \frac{1}{2}R$ and thus, by best-response to β_1 in ω_2, $\sigma_1(\omega_2) = U$. But the best-response to U in ω_2 is R, not $L = \sigma_2(\omega_2)$.

Assume next by contradiction that $\sigma_2(\omega_2) = R$, then $\beta_1(\alpha) = R$ and thus, by best-response to β_1 in ω_2, $\sigma_1(\omega_2) = D$. But, the best-response to D in ω_2 is L, not $R = \sigma_2(\omega_2)$.

in ω_2, it should be that $\beta_1(\alpha) = \frac{1}{2+x}L \oplus \frac{1+x}{2+x}R$, thereby implying (to satisfy the consistency condition) that the Column player is choosing $\sigma_2(\omega_2) = \frac{2}{2+x}L \oplus \frac{x}{2+x}R$. For $x < 1$, it is then readily verified that $d(\sigma_2(\omega_2), \sigma_2(\omega_1)) < d(\sigma_2(\omega_2), \beta_1(\alpha))$ (given that $\sigma_2(\omega_1) = L$ and $1 - \frac{2}{2+x} = \frac{x}{2+x} < \frac{1}{2+x} = \frac{2}{2+x} - \frac{1}{2+x}$) and thus the local calibration condition is violated for game ω_2 (it would have to be re-assigned to the analogy class $\alpha' = \{\omega_1\}$ instead of $\alpha = \{\omega_2, \omega_3\}$.

We have shown that there is no locally calibrated ABEE when $x < 1$, which of course implies that there is no globally calibrated ABEE in this case too.

When $1 \leq x$, the analogy partition with $\alpha = \{\omega_2, \omega_3\}$ as the non-singleton analogy class is the only possibility for a locally calibrated ABEE and thus the only candidate for a globally calibrated ABEE. However, when $x < 2$, global calibration would lead to put together ω_1 and ω_2 given that $\sigma_2(\omega_1) = L$, $\sigma_2(\omega_2) = \frac{2}{2+x}L \oplus \frac{x}{2+x}R$ and $\sigma_2(\omega_3) = R$, thereby invalidating the global calibration condition in this case.**Q.E.D.**

Comments. 1) If we were to consider the Kullback-Leibler divergence, the analogy partition with $\alpha = \{\omega_2, \omega_3\}$ as the non-singleton analogy class for player 1 would lead to a locally but not globally calibrated ABEE in the above example for all $x < 2$.[12] This observation illustrates how different conclusions can be obtained whether the Euclidean distance or the Kullback-Leibler divergence is considered.

2) In Example 2, the only Nash equilibrium in ω_2 employs mixed strategies. One may wonder whether the inexistence of a pure Nash equilibrium in at least one game ω is a required property for the inexistence of a locally calibrated ABEE. To address this, consider a setting in which there is a pure Nash equilibrium in each normal form game. It is readily verified that when K_i is no smaller than the number of actions employed by player j across the various pure Nash Equilibria obtained when the game $\omega \in \Omega$ is varied, the Nash strategy profiles can be played in a globally calibrated ABEE by considering for player i an analogy partition An_i that assigns games ω with the same Nash action of player j to the same analogy class. However, when K_i is smaller than this number for at least one

[12] For the local calibration part, observe that with the Kullback-Leibler divergence, the mixed behavior $\sigma_2(\omega_2) = \frac{2}{2+x}L \oplus \frac{x}{2+x}R$ would be better explained by $\beta_1(\alpha) = \frac{1}{2+x}L \oplus \frac{1+x}{2+x}R$ than by the pure theory L that could not generate R observations (even if arbitrarily rare).

Global calibration though would still require that game ω_2 be assigned together with ω_1 whenever $x < 2$.

player i, then there may be no locally calibrated ABEE. We illustrate this in the online appendix (using a three game setting with $d(p_j, p'_j) = \big(E(p_j) - E(p'_j)\big)^2$ and identifying actions with points on the real line).

3) A globally calibrated ABEE should not be confused with the outcome of a two-stage game in which in stage 1, players 1_C and 2_C would simultaneously choose the analogy partitions of players 1 and 2 who would play a corresponding ABEE in stage 2 with an objective of player i_C as given by (1). The main difference between the two approaches is the commitment aspect in the two stage version which would allow player i_C to choose an analogy partition that is not optimal with respect to (1) given the actual play of player i in stage 2.

2.5 Distributional calibrated analogy-based expectation equilibrium

We propose getting around the existence problem by adopting a distributional approach (that will be interpreted after the definitions are in place). Formally, we allow the analogy partition An_i of player i to take different realizations in $\mathcal{K}_i$, and we refer to λ_i as the distribution of An_i over $\mathcal{K}_i$. The distributions of analogy partitions of the two players are viewed as independent of one another (formalizing a random assignment assumption, see the learning dynamics described below for elaborations). We refer to $\lambda = (\lambda_i, \lambda_j)$ as the profile of these distributions, and we let $\Lambda = \Delta\mathcal{K}_i \times \Delta\mathcal{K}_j$ be the set of (λ_i, λ_j). For each analogy partition An_i of player i in the support of λ_i referred to as $Supp\lambda_i$, we let $\sigma_i(\cdot \mid An_i) : \Omega \to \Delta A_i$ refer to the mapping describing how player i with analogy partition An_i behaves in the various games $\omega \in \Omega$. We refer to $\sigma_i = (\sigma_i(\cdot \mid An_i))_{An_i \in Supp\lambda_i}$ as player i's strategy, and we let $\sigma = (\sigma_i, \sigma_j)$ denote the strategy profile, the set of which is still denoted Σ.

Given $\lambda \in \Lambda$ and $\sigma \in \Sigma$, we can define the aggregate behaviors of the two players in each game, as aggregated over the various realizations of analogy partitions. We have in mind that these aggregate behaviors in the various games $\omega \in \Omega$ constitute the only data accessible to players, thereby implying that only these aggregates are used to construct the analogy-based expectations and implement the clustering. Formally, the aggregate strategy of player j in game ω is given by

$$\overline{\sigma}_j(\omega) = \sum_{An_j \in \mathcal{K}_j} \lambda_j(An_j)\sigma_j(\omega \mid An_j). \tag{2}$$

Let $\bar{\sigma} = (\bar{\sigma}_i, \bar{\sigma}_j)$ denote a profile of aggregate strategies and let $\overline{\Sigma}$ denote the set of such profiles.

The analogy-based expectation of player i defines for each analogy partition $An_i \in Supp\lambda_i$ and each analogy class $\alpha_i \in An_i$, the aggregate behavior of player j in α_i denoted by $\beta_i(\alpha_i \mid An_i) \in \Delta A_j$ (the dependence on An_i is here to stress that player i with analogy partition An_i considers only the aggregate behaviors in the various analogy classes in An_i). Similarly as above, $\beta_i(\cdot \mid An_i)$ is said to be consistent with $\overline{\sigma}_j$ iff, for all $\alpha_i \in An_i$,

$$\beta_i(\alpha_i \mid An_i) = \sum_{\omega \in \alpha_i} p(\omega)\overline{\sigma}_j(\omega) / \sum_{\omega \in \alpha_i} p(\omega). \tag{3}$$

We are now ready to propose the distributional extensions of our previous definitions.

Definition 4. *Given $\lambda = (\lambda_i, \lambda_j) \in \Lambda$, a strategy profile $\sigma = (\sigma_i, \sigma_j) \in \Sigma$ is a distributional analogy-based expectation equilibrium (ABEE) iff there exists $\beta = (\beta_i, \beta_j)$ such that for every player i and $An_i \in Supp\lambda_i$, we have that i) $\sigma_i(\cdot \mid An_i)$ is a best-response to $\beta_i(\cdot \mid An_i)$ and ii) $\beta_i(\cdot \mid An_i)$ is consistent with $\overline{\sigma}_j$ (where $\overline{\sigma}_j$ is derived from σ_j as in (2)).*

Definition 5. *A pair $(\sigma, \lambda) \in \Sigma \times \Lambda$ is a locally (resp. globally) calibrated distributional analogy-based expected equilibrium iff i) σ is a distributional ABEE given λ, and ii) for every player i and $An_i \in Supp\lambda_i$ (where $\lambda = (\lambda_i, \lambda_j)$), An_i is locally (resp. globally) calibrated with respect to $\overline{\sigma}_j$ (where $\overline{\sigma}_j$ is derived from σ_j as in (2)).*

Clearly, a calibrated distributional ABEE coincides with a calibrated ABEE if the distributions of analogy partitions assign probability 1 to a single analogy partition for both players i and j. Calibrated distributional ABEE are thus generalizations of calibrated ABEE. We now establish an existence result.

Theorem 1. *In finite environments, there always exists a locally (resp. globally) calibrated distributional ABEE when d is the square of the Euclidean distance or the Kullback-Leibler divergence.*

To prove this result we focus on the existence of a globally calibrated distributional ABEE, since any globally calibrated distributional ABEE is obviously a locally calibrated distributional ABEE. By Kakutani fixed point theorem, we show that there is a fixed point of a compound correspondence that maps $\bar{\Sigma} \times \Lambda$ into itself. This correspondence is defined so that any fixed point of it is a globally calibrated distributional ABEE.

More precisely, the first aspect of the correspondence is the function mapping $(\overline{\sigma}, \lambda)$ into the analogy-based expectations $\beta = (\beta_i, \beta_j)$ for the various analogy partitions in the support of λ where such analogy-based expectations β are required to be consistent with $\overline{\sigma}$. Given (3), this function is obviously continuous. From β, one can define the best-response correspondences for each of the analogy partitions in the support of λ, thereby defining for $i = 1, 2$ sets of best-responses $\sigma_i(\cdot \mid An_i)$ (to $\beta_i(\cdot \mid An_i)$) for the various An_i in the support of λ_i. Such best-response correspondences are upper-hemicontinuous as in the standard case. Combining with λ according to (2), this gives rise to sets of $\overline{\sigma}$, and it is readily verified that this part of the correspondence satisfies the upper-hemicontinuity condition required for Kakutani's theorem.

The second aspect of the correspondence concerns the one mapping $\overline{\sigma}$ into the distributions over analogy partitions that would be globally calibrated with respect to $\overline{\sigma}$. This correspondence defines a convex hull with extreme points given by the solutions to (1). This correspondence satisfies the upper-hemicontinuity conditions required for Kakutani's theorem when we consider the squared Euclidean distance (variance criterion). For the Kullback-Leibler divergence criterion, some extra care is needed as $d(q, q')$ can diverge to infinity when $supp[q] \nsubseteq supp[q']$. We deal with this by making extra use of the consistency requirement. Details about this and the overall proof appear in the Appendix.

Example 1 (continued) For $x < 2$ and whether d is the square of the Euclidean distance or the Kullback-Leibler divergence, there exists a unique globally calibrated distributional ABEE. In any such ABEE, it should be that $\sigma_2(\omega_2) = \frac{1}{2}L \oplus \frac{1}{2}R$ so that for the purpose of global calibration, one can equally have $An = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$ or $An' = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$ (given that $\sigma_2(\omega_1) = L$ and $\sigma_2(\omega_3) = R$).[13] Given this, consistency implies that $\beta_1(\{\omega_1, \omega_2\} \mid An) = \frac{3}{4}L \oplus \frac{1}{4}R$ and thus $\sigma_1(\omega_2 \mid An) = U$. Similarly, $\beta_1(\{\omega_2, \omega_3\} \mid An') = \frac{1}{4}L \oplus \frac{3}{4}R$ and thus

[13]For local calibration, there is extra degree of freedom as any $\sigma_2(\omega_2) = qL \oplus (1 - q)R$ with $q \in [1/3, 2/3]$ could be used.

$\sigma_1(\omega_2 \mid An') = D$ for all $x < 2$. In order to let the Column player be indifferent between L and R in ω_2, it should be that $\overline{\sigma}_1(\omega_2) = \frac{1}{2}U \oplus \frac{1}{2}D$, thereby implying that $\lambda_1 = \frac{1}{2}An \oplus \frac{1}{2}An'$, i.e., the Row player should randomize 50:50 between the two analogy partitions An and An'. Completing this with the best-response of player 1 in game $\omega_1$to $\frac{3}{4}L \oplus \frac{1}{4}R$ (resp. L) when using An (resp. An'), and the best-response of player 1 in game ω_3 to R (resp. $\frac{1}{4}L \oplus \frac{3}{4}R$) when using An (resp. An') provides a complete description of the globally calibrated distributional ABEE.

Comment. In a calibrated distributional ABEE, it is the case that the same datapoint corresponding to the same game may be assigned to different clusters/analogy classes depending on the analogy partition. More generally, one may wonder if two games corresponding to datapoints which are nearby would always be assigned to the same analogy class. Clearly, this is not so in a calibrated distributional ABEE (as just discussed). By contrast, in a globally calibrated ABEE, if players' incentives to follow their strategies are strict in each game and if there is a unique solution to the clustering problem (as would arise for generic values of p), then games that would correspond to the same equilibrium behavior would have to belong to the same analogy class. This suggests why some mixing is needed for this to arise.

2.6 Learning foundation

In this part, we introduce learning dynamics involving populations of players in the roles of $i = 1, 2$ the steady states of which correspond to the globally calibrated distributional ABEE.

Learning dynamics. There is a continuum of mass 1 of subjects assigned to the role of player $i = 1, 2$. We refer to the subjects assigned to the role of player i as population i. In each period, subjects from populations i and j are randomly matched to play a randomly selected game ω. Throughout this Section, we assume that the share of game ω that is being played matches the prior probability $p(\omega)$ that game ω is chosen.

We introduce two perturbations that are used to deal with possible indifferences (both at the stage when strategies are chosen and at the stage when categorizations are chosen). First, when playing a game, we assume the payoffs are slightly perturbed, as is commonly considered in the learning literature (see in particular Fudenberg and Kreps (1993) or more recently Esponda and Pouzo (2016)). Formally, let $\widetilde{\rho}_i$ be a random variable with a continuous density g_i on $[0, 1]$ and $\varepsilon > 0$ a number that should be thought of as small (ε measures the degree of perturbation). We assume that before playing game ω, player i attaches an extra payoff $\varepsilon\rho_i(a_i, \omega)$ to action a_i as compared with the baseline payoff $u_i(a_i, a_j, \omega)$ defined in the main model where $\rho_i(a_i, \omega)$ is a realization drawn from $\widetilde{\rho}_i$ and the draws are assumed to be independent across actions a_i and games ω. As before, we assume that the distribution of realizations in the population matches the densities and probabilities induced by p and $\widetilde{\rho}_i$. To be more specific, player i in game ω (with draws $\rho_i(a_i, \omega), \rho_i(a'_i, \omega)$) picks action a_i whenever for all $a'_i \neq a_i$,[14]

$$u_i(a_i, \beta_i, \omega) + \varepsilon\rho_i(a_i, \omega) > u_i(a'_i, \beta_i, \omega) + \varepsilon\rho_i(a'_i, \omega)$$

where β_i refers here to player i's expectation about player j's behavior in ω.

The second perturbation concerns how clustering is implemented. Suppose in the previous period $\overline{\sigma}_j(\omega)$ represents the aggregate play of population j when playing game ω. We assume that subjects in population i before knowing the game ω, implement the clustering of the corresponding datapoints, but instead of clustering $(\overline{\sigma}_j(\omega))_{\omega\in\Omega}$ they consider a slight perturbation of these datapoints (where the perturbed datapoints can be thought of as being the result of measurement errors). Formally, let $\widetilde{\eta}_i$ be a random vector with continuous density h_i over the interior of ΔA_j. We assume that a given player of population i implements a (global) clustering into K_i classes of $\left(\overline{s}_j(\omega) \equiv \frac{\overline{\sigma}_j(\omega)+\varepsilon\eta_i(\omega)}{1+\varepsilon}\right)_{\omega\in\Omega}$ where $\eta_i(\omega)$ is a realization drawn from $\widetilde{\eta}_i$ and the draws are assumed to be independent across games ω and across subjects. As before, we assume that the distributions of realizations in the population match the density $\widetilde{\eta}_i$. To be more specific, player i (with draws $\eta_i(\omega)$) picks the partitioning into K_i classes so as to solve[15]

[14]Cases of indifference are insignificant whenever $\widetilde{\rho}_i$ is distributed in the continuum as assumed here.

[15]For generic $\eta_i(\omega)$, there is a unique solution, thus the handling of indifferences is inconsequential when $\widetilde{\eta}_i$ has a density with no atom, as assumed here.

$$\arg\min_{P_i \in \mathcal{K}_i} \sum_{c_i \in P_i} p(c_i) \sum_{\omega \in c_i} p(\omega \mid c_i) d(\overline{s}_j(\omega), \beta_i(c_i))$$

where $\beta_i(c_i)$ is here the mean of $\overline{s}_j(\omega)$ conditional on $\omega \in c_i$.

The learning dynamics is described as follows. Given the aggregate plays $\overline{\sigma}^{t-1}(\omega)$ in period $t-1$, subjects of population i in period t implement the optimal clustering with respect to the subject-specific perturbed datapoints induced by $\left(\overline{\sigma}_j^{t-1}(\omega)\right)_{\omega \in \Omega}$ as just explained. This fixes for each subject in the population of player i a belief about how a_j is chosen in game ω. More precisely, for each game in cluster c_i, the belief is identified with the mean of $\overline{s}_j(\omega)$ conditional on $\omega \in c_i$, which corresponds to the representative point in cluster c_i found at the clustering stage.[16] Then players are randomly matched to play the various games, and they choose a perturbed best-response in the game they are assigned to as given by their expectations β_i and the realized perturbations $\rho_i(a_i, \omega)$. Integrating over the various subjects, this generates new aggregate data $\overline{\sigma}^t(\omega)$ for the various games ω in period t. The dynamics is then fully pinned down by the initial values of $\overline{\sigma}^0(\omega)$ used in period 1 (as well as ε, g_i, h_i).

Steady state. We first establish that for a fixed ε, there always exists a steady state of the learning dynamics just described. We next establish that the limits of such steady states as ε converges to 0 correspond to the globally calibrated distributional ABEE. The proofs appear in the Appendix.

Proposition 2. *For a fixed ε, there always exists a steady state of the learning dynamics.*

Proposition 3. *Consider a sequence of steady states $(\sigma^{(\varepsilon)}, \lambda^{(\varepsilon)})$ of the learning dynamics induced by ε where $\sigma^{(\varepsilon)}$ denotes the ex ante strategy (prior to the realizations of the perturbations ρ) and $\lambda^{(\varepsilon)}$ denotes the distribution of the profile of analogy partitions.*[17] *Consider an accumulation point (σ, λ) of $(\sigma^{(\varepsilon)}, \lambda^{(\varepsilon)})$ as ε*

[16] This is the extra place where the result of Lemma 1 is being used, in the sense that the representative points found at the clustering stage can directly be used as expectations at the strategy selection stage. Note that this also applies to the setting discussed in Section 3 (since there strategies can be reduced to their expectations both at the clustering stage and at the best-response stage), but not to the example discussed in the online appendix.

[17] $\sigma_i^{(\varepsilon)}$ is a strategy of player i that depends on the game ω and the analogy partition An_i of player i as in the general construction above.

tends to 0. (σ, λ) *is a globally calibrated distributional ABEE.*

Discussion. As already mentioned, the perturbation of payoffs is similar to that considered in Fudenberg and Kreps (1993), and it allows us to ensure that the randomization that may arise when players employ mixed strategies are made independently across players. The perturbation of $\overline{\sigma}$ at the clustering stage is new to the present framework, but it serves a similar purpose of ensuring that the distributions of analogy partitions are independent across players. We have chosen to formulate results in terms of steady states and limits of those as the magnitude of the perturbations (parameterized by ε) vanishes so as to strengthen the link to the previously introduced solution concept. This is a bit different from Fudenberg and Kreps (1993) or Esponda and Pouzo (2016) who consider fixed perturbations and show properties of limit strategies of the learning dynamics when these are assumed to be converging. Fixing ε, we could establish that if there is convergence, it must correspond to a steady state of the learning dynamics. This would require relying on some form of the law of large numbers similarly as in Fudenberg and Kreps (1993) or Esponda and Pouzo (2016).

The construction of the learning dynamics has been made using global calibration. One could instead assume that at the clustering stage, players rely on the K-means algorithm with the perturbed data, and we would then have to specify which initial conditions are used when implementing the algorithm (possibly allowing different subjects to use different initial conditions). This would induce potentially extra complications (compared to the above analysis) as it would require deriving extra properties regarding how the K-means clustering algorithm transforms perturbed datasets into analogy partitions. We leave the analysis of this for future research.

3 Strategic interactions with linear best-replies

In this section we apply the notion of Calibrated ABEE to families of games with continuous action spaces parameterized by an interaction parameter μ, which takes values in an interval of the real line. This parameter is a determinant of the intensity of players' reactions to their opponent's behavior. Players have best-responses which are linear both in the strategy of the opponent and in μ.

Formally, we consider a family of games parameterized by $\mu \in [-1, 1]$, where μ is distributed according to a continuous density function f with cumulative denoted by F. Players observe the realization of μ and player $i = 1, 2$ chooses action $a_i \in \mathbb{R}$. In game μ, when player i expects player j to play according to $\sigma_j \in \Delta\mathbb{R}$, player i's best-response is:

$$BR_i(\mu, \sigma_j) = A + \mu B + \mu C E(\sigma_j),$$

where $E(\sigma_j)$ denotes the mean action derived from the distribution σ_j, and A, B and C are constants with $0 < C < 1$. We will analyze separately the cases in which $\mu \in [0, 1]$ and $\mu \in [-1, 0]$, and in each case we will assume that $f(\cdot)$ has full support. In the former case, the games exhibit strategic complementarity. In the latter, they exhibit strategic substitutability.

The restriction to linear best-replies while demanding in some respects allows us to illustrate in a simple way the implications of our general framework. It also allows us to accommodate classic applications.[18]

In particular, consider the case of strategic complementarity ($\mu \geq 0$). A game with linear best-responses arises in a duopoly with differentiated products in which firms have constant marginal costs, demand is linear, and firms compete in prices à la Bertrand (see Vives 1999 for a textbook formulation). Alternatively, this framework can capture a reduced form of moral hazard in team problems (a specific formulation of the model introduced by Holmström 1982) in which the agents receive a bonus if the team is successful, agents simultaneously choose how much effort to exert, the probability of success depends on the profile of effort in a bilinear way and the cost of effort is quadratic.[19]

Consider next the case when μ is non-positive so that the game exhibits strategic substitutability. A setting fitting our formulation is one of a duopoly with differentiated products with constant marginal costs and linear demands, but this time assuming firms compete in quantities à la Cournot (see again Vives 1999 for

[18] It may be mentioned that in our formulation, we allow the actions to take any value (positive or negative) whereas in some of the applications mentioned below it would be natural to impose that the actions (quantities, prices or effort level) be non-negative. We do not impose non-negativity constraints to avoid dealing with corner solutions, but none of our qualitative insights would be affected with such additional constraints.

[19] Complementarity is obtained for positive coefficients applying to the product of effort levels in the probability of success.

elaborations).

Regardless of the sign of μ, it is readily verified that there exists a unique Nash Equilibrium of the game with parameter μ. It is symmetric, it employs pure strategies and it is characterized by $a_1^{NE}(\mu) = a_2^{NE}(\mu) = \frac{A+\mu B}{1-\mu C}$. The function $a_i^{NE}(\mu)$ is continuous and monotone in μ. When $B = -AC$, the function $a_i^{NE}(\mu)$ is flat. The function is strictly increasing (decreasing) and convex (concave) in μ for B greater (smaller) than $-AC$, when $\mu \in [-1, 1]$.

In our family of games parameterized by μ, it is natural to impose that if two games μ^* and μ^{**} are bundled together in the same analogy class, any game μ with μ in between μ^* and μ^{**} should also be bundled with μ^* and μ^{**} as well.[20] Accordingly, we will be considering analogy partitions with the property that each analogy class is an interval of μ, and we will refer to these as *interval analogy partitions.*

Specifically, assume that players use (pure) symmetric interval analogy partitions, splitting the interval into K subintervals, so that

$$An_1 = An_2 = \{[\mu_0, \mu_1], (\mu_1, \mu_2], \ldots, (\mu_{K-1}, \mu_K]\}$$

where $\mu_0 = 0$, $\mu_K = 1$ in the case of strategic complements, and $\mu_0 = -1, \mu_K = 0$ in the case of strategic substitutes.[21]

Since analogy partitions are symmetric, we simplify notation by dropping the subscript that indicates whether player 1 or 2 is considered. We simply denote the interval $(\mu_{k-1}, \mu_k]$ by α_k for $k = 2, ...K - 1$ and $\alpha_1 = [\mu_0, \mu_1]$.

Our general framework as introduced in Section 2 considered the finite case in which the action and the state spaces are both finite. In general, extending the definitions of equilibrium and deriving existence results when either of these spaces lies in the continuum can raise difficulties.[22] However, in the present context in which best-responses are linear, there is an easy way of extending the definitions of analogy-based expectation equilibrium and consistency restricting attention to the

[20] Such a desideratum may apply more broadly as soon as there is a natural notion of proximity between games independently of how players behave.

[21] Whether μ_k is assigned to (μ_{k-1}, μ_k) or (μ_k, μ_{k+1}) plays no role in our setting with a continuum of μ.

[22] A similar observation applies to other solution concepts such as the sequential equilibrium as recently illustrated in Myerson and Reny (2020).

mean action of the opponent as opposed to the entire distribution. Also, in such a case it is natural to compare the behaviors in different games using the Euclidean distance between the mean action these games induce, and we will accordingly consider the square of the Euclidean distance in the space of these mean actions for clustering purposes.

Specifically, with some abuse of notation, we will refer to $\beta_i(\alpha_k)$ as the expected mean action of player j in the analogy class α_k. The consistency of β_i with σ_j imposes that $\beta_i(\alpha_k) = \frac{1}{F(\mu_k)-F(\mu_{k-1})} \int_{\mu_{k-1}}^{\mu_k} \sigma_j(\mu) f(\mu) d\mu$, where $\sigma_j(\mu)$ denotes the (mean) action chosen by player j in game μ. Moreover in each game $\mu \in \alpha_k$, best-response requires that player i chooses action $BR_i(\mu, \beta_i(\alpha_k)) = A + \mu(B + C\beta_i(\alpha_k))$ as given by μ and his analogy-based expectation $\beta_i(\alpha_k)$ about the mean action in α_k.

Given a (symmetric) interval analogy partition profile $An_1 = An_2 = \{[\mu_0, \mu_1], \ldots, (\mu_{K-1}, \mu_K]\}$, an ABEE is a strategy profile (σ_1, σ_2) such that for each player i, each class α_k and each game $\mu \in \alpha_k$, we have $\sigma_i(\mu) \in BR_i(\mu, \beta_i(\alpha_k))$ with the requirement thate β_i is consistent with σ_j. Exploiting the linearity of the best-response, it is easily established (through routine calculations provided in the online Appendix) that there exists a unique ABEE, which is symmetric, whatever the (symmetric) interval analogy partition.

Proposition 4. *Assume players use symmetric interval analogy partitions. There exists a unique ABEE where, for all $k = 1, \ldots, K$, $\beta_1(\alpha_k) = \beta_2(\alpha_k) = \frac{A+B\mathbb{E}[\mu|\alpha_k]}{1-C\mathbb{E}[\mu|\alpha_k]}$ and for $\mu \in \alpha_k$, $\sigma_1(\mu) = \sigma_2(\mu) = A + \mu \frac{B+AC}{1-C\mathbb{E}[\mu|\alpha_k]}$.*

Since under symmetric interval analogy partitions the ABEE is symmetric, we drop the subscript that refers to players and we write $\beta_1(\alpha_k) = \beta_2(\alpha_k) = \beta(\alpha_k)$. We also let $a(\mu|\alpha_k)$ refer to $A + \mu \frac{B+AC}{1-C\mathbb{E}[\mu|\alpha_k]}$ for the remainder of this section where as seen in Proposition 4, $a(\mu|\alpha_k) = A + \mu \frac{B+AC}{1-C\mathbb{E}[\mu|\alpha_k]}$ describes the ABEE strategies of players 1 and 2 in the analogy class α_k. The function $a(\mu|\alpha_k)$ is linear in μ. Similarly to the discussion of Nash Equilibrium above, when $B = -AC$, the function $a(\mu|\alpha_k)$ is flat, and it is strictly increasing (decreasing) in μ for B greater (smaller) than $-AC$, if $\mu \in [0, 1]$.

As far as clustering is concerned, and as already mentioned, we consider the square of the Euclidean distance in the mean actions. That is, considering a symmetric interval analogy partition given by $\{\alpha_k\}_{k=1}^K$ and the associated ABEE (as described by $\beta(\alpha_k)$ and $a(\mu \mid \alpha_k)$), $\{\alpha_k\}_{k=1}^K$ is locally calibrated, if for all $k = 1, \ldots, K$ and all $\mu \in \alpha_k$,[23]

$$(\beta(\alpha_k) - a(\mu|\alpha_k))^2 \leq (\beta(\alpha_{k'}) - a(\mu|\alpha_k))^2, \forall k' \neq k.$$

By the monotonicity of the function $a(\mu|\alpha_k)$ the problem of local calibration boils down to verifying the above inequalities only at the extreme points of each analogy class. That is, the sequence $\{\mu_0, \mu_1, \ldots, \mu_K\}$ generates an interval analogy partition that is locally calibrated with respect to the corresponding ABEE, if and only if, for $k = 1, \ldots, K-1$,

$$(\beta(\alpha_k) - a(\mu_k|\alpha_k))^2 \leq (\beta(\alpha_{k+1}) - a(\mu_k|\alpha_k))^2$$

and

$$(\beta(\alpha_{k+1}) - a(\mu_k|\alpha_{k+1}))^2 \leq (\beta(\alpha_k) - a(\mu_k|\alpha_{k+1}))^2 \tag{4}$$

As far as global calibration is concerned, one has to check for a given candidate interval analogy partition $An = \{\alpha_k\}_{k=1}^K$ whether

$$An = \arg \min_{\{\alpha'_k\}_{k=1}^K} \sum_k \int_{\alpha'_k} \left[\beta(\alpha'_k) - a^{ABEE}(\mu)\right]^2 f(\mu) d\mu$$

where $a^{ABEE}(\mu) \equiv A + \mu \sum_{k=1}^K 1_{\{\mu_{k-1} < \mu \leq \mu_k]\}} \frac{B+AC}{1-C\mathbb{E}[\mu|\alpha_k]}$ is the ABEE strategy given An and $\beta(\alpha'_k) = E(a^{ABEE}(\mu) \mid \mu \in \alpha'_k)$.

3.1 Strategic Complements

In this part we assume that μ is distributed according to a continuous density f with support on $[0, 1]$.

Given $(\mu_k)_{k=0}^K$, we make the important observation that

$$a^{ABEE}(\mu) = A + \mu \sum_{k=1}^K 1_{\{\mu_{k-1} < \mu \leq \mu_k]\}} \frac{B + AC}{1 - C\mathbb{E}[\mu|\alpha_k]}$$

[23] Clearly, for local calibration, we could consider the Euclidean distance instead of the square of the distance. This is so because comparisons are only in terms of the mean action, which is one-dimensional.

has discontinuities at $\mu_1, \mu_2, \ldots, \mu_{K-1}$. If $B \geq -AC$, the function $a^{ABEE}(\mu)$ is increasing in μ and the discontinuities take the form of upward jumps. Similarly, if $B < -AC$, the function $a^{ABEE}(\mu)$ is decreasing in μ and the discontinuities take the form of downward jumps. The direction of the jumps is a consequence of the strategic complement aspect, and it will play a key role in the analysis of local calibration. Indeed assuming $B \geq -AC$, as one moves in the neighborhood of μ_k from the analogy class $(\mu_{k-1}, \mu_k]$ to the analogy class $(\mu_k, \mu_{k+1}]$, the perceived mean action of the opponent jumps upwards and this leads to an upward jump in the best-response.

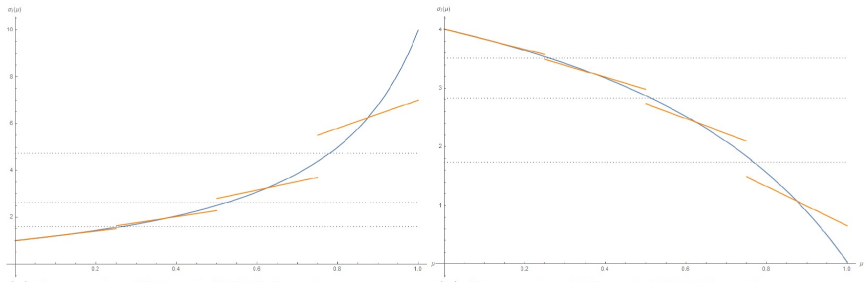
There is a simple geometric characterization of local calibration. Assuming $B \geq -AC$, we have that $\beta(\alpha_k) \leq \beta(\alpha_{k+1})$, for all k. The local calibration requirements summarized by inequalities in (4) are equivalent to the condition that the arithmetic average of the analogy-based expectations of two adjacent analogy classes should be between the largest action in the first and the smallest action in the second analogy class. That is,

$$a(\mu_k|\alpha_k) \leq \frac{\beta(\alpha_k) + \beta(\alpha_{k+1})}{2} \leq a(\mu_k|\alpha_{k+1}).$$

Similarly, when $B < -AC$, the ABEE function is strictly decreasing and (4) can be reduced to $a(\mu_k|\alpha_k) \geq \frac{\beta(\alpha_k)+\beta(\alpha_{k+1})}{2} \geq a(\mu_k|\alpha_{k+1})$. These inequalities can receive a simple graphical interpretation as illustrated in Figure 1 where the horizontal dashed lines in black represent the arithmetic average between the analogy-based expectations of two consecutive classes, and whenever $a^{ABEE}(\mu)$ does not cross any dashed line, the requirements for local calibration are satisfied by that analogy class.[24]

[24]Figure 1 shows how the (simplified) local calibration requirements would appear graphically. There are two graphs, one for $B > -AC$ on the left and one for $B < -AC$ on the right. Both graphs depict how the Nash Equilibrium function $a^{NE}(\mu) = \frac{A+\mu B}{1-\mu C}$ (in blue) and the ABEE function $a^{ABEE}(\mu)$ (in orange) change as μ varies. For these graphs we assume that μ is distributed uniformly over $[0, 1]$, we let $K = 4$, and we pick the interval analogy partition induced by the equal splitting sequence $\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$. When $B > (<)AC$, $a^{NE}(\mu)$ and $a^{ABEE}(\mu)$ are strictly increasing (decreasing).

Figure 1: Graphical Illustration of Local Calibration Requirements for $K = 4$ and μ-sequence such that $\mu_k = \frac{k}{4}$, $k = 0, 1, \ldots, 4$



(a) Increasing NE and ABEE functions
$A = 1$, $B = 1$ and $C = 0.8$

(b) Decreasing NE and ABEE functions
$A = 4.01$, $B = -4$ and $C = 0.6$

One can easily see from Figure 1 that the analogy partitions depicted in the graphs are locally calibrated. As a matter of fact, and as we will show later, when μ is uniformly distributed between 0 and 1, an analogy partition that splits the interval into K subintervals of equal size leads to a locally calibrated ABEE. For more general distributions, we introduce the notion of *equidistant-expectations sequence* $\mu_0, \mu_1, \ldots, \mu_K$ defined so that for any μ_k, with $k \neq 0, 1$, the Euclidean distance between μ_k and the mean value of μ in $(\mu_{k-1}, \mu_k]$ is equal to the Euclidean distance between μ_k and the mean value of μ in $(\mu_k, \mu_{k+1}]$. That is, $\mu_k - \mathbb{E}[\mu|(\mu_{k-1}, \mu_k]] = \mathbb{E}[\mu|(\mu_k, \mu_{k+1}]] - \mu_k$. We refer to the corresponding interval partition $(\alpha_k)_{k=1}^K$ with $\alpha_k = (\mu_{k-1}, \mu_k]$ as the *equidistant-expectations partition.* We note that when μ is uniformly distributed on $[0, 1]$, the equidistant-expectations sequence is uniquely defined by $\mu_k = \frac{k}{K}$, and in this case we refer to it as the equal splitting sequence. For more general density functions f, it is readily verified (by repeated application of the intermediate value theorem) that:[25]

Lemma 2. *There always exists at least one equidistant-expectations partition.*

3.1.1 Locally calibrated ABEE

Proposition 5. *In the environment with strategic complements, consider an equidistant-expectations partition An, and let $a^{ABEE}(\mu)$ be the corresponding ABEE. (a^{ABEE}, An) is a locally calibrated ABEE.*

[25]See the online appendix for details. We also conjecture that there may be multiple such sequences in some cases. Toward this end, consider the pdf $f(\mu)$ being ε in $[0, \frac{1}{2}]$ and $2 - \varepsilon$ in $(\frac{1}{2}, 1]$. As $\varepsilon \to 0$, two sequences would satisfy the property: $\{0, 1/2, 1\}$ and $\{0, 3/4, 1\}$. This discontinuous pdf is excluded by our assumptions, but we conjecture that we can construct densities with continuous pdf with the same property.

The rough intuition for this result can be understood as follows. Suppose we were considering games with no interaction term, i.e., such that $C = 0$. Then in game μ, players would be picking their dominant strategy $a(\mu) = A+\mu B$ irrespective of the analogy partition, given that players would not care about the action chosen by their opponent. It is readily verified that the K-means clustering of the points $a(\mu)$ would lead to pick an *equidistant-expectations partition* in this case, and this would force a locally calibrated ABEE when $C = 0$ to be relying on such *equidistant-expectations partitions.* Allowing for non-null interaction parameters C makes the problem of finding a locally calibrated ABEE a priori non-trivial due to the endogeneity of the data generated by the ABEE with respect to the chosen analogy classes, as explained in Section 2. However, what the Proposition establishes is that using the same analogy classes as those obtained when $C = 0$ can be done to construct a locally calibrated ABEE. Intuitively, this is so because the strategic complement dimension makes the points obtained through ABEE in a given class of the equidistant-expectations partition look closer to one another relative to points outside a class, as compared with the case in which $C = 0$. As a result, the local calibration conditions which hold for the equidistant-expectation partition when $C = 0$ hold a fortiori when C is non-null. This rough intuition is confirmed in the Appendix.

Building on Proposition 5, one may seek to characterize the set of interval partitions that can arise in locally calibrated ABEE. Clearly, not all interval partitions can arise as for example with $K = 2$ if μ_1 is either too close to 0 (resp. 1), the game μ slightly above (resp. below) μ_1 would not satisfy the local calibration requirement. The following proposition characterizes the set of interval partitions that can arise in locally calibrated ABEE.

Proposition 6. *Consider the strategic complements environment. Let An be a pair of symmetric interval analogy partitions generated by the increasing sequence $\{\mu_0, \mu_1, \ldots, \mu_K\}$ with corresponding ABEE a^{ABEE}. (a^{ABEE}, An) is a locally calibrated ABEE if and only if the following conditions hold:*

(i) $\mu_0 = 0$ and $\mu_K = 1$;

(ii) given (μ_{k-1}, μ_k), μ_{k+1} satisfies $\mathbb{E}[\mu|(\mu_k, \mu_{k+1}]] \geq \frac{2\mu_k - \mathbb{E}[\mu|(\mu_{k-1},\mu_k]]}{1+2C(\mu_k - \mathbb{E}[\mu|(\mu_{k-1},\mu_k]])}$, for $k = 1, \ldots, K-1$;

(iii) given (μ_{k-1}, μ_k), if $\mathbb{E}[\mu|(\mu_{k-1}, \mu_k]] < \frac{1}{2C}$, then μ_{k+1} satisfies

$$\mathbb{E}[\mu|(\mu_k, \mu_{k+1}]] \leq \mu_k + \frac{\mu_k - \mathbb{E}[\mu|(\mu_{k-1},\mu_k]]}{1-2C\mathbb{E}[\mu|(\mu_k,\mu_{k-1}]]} \; for \; k = 1, \ldots, K-1.$$

Proof. First, note that condition (i) simply requires that the sequence starts at 0 and ends at 1, so that $\{\mu_k\}_{k=1}^K$ generates in fact an analogy partition. Hence, it is a necessary condition. Due to the monotonicity of a^{ABEE}, an analogy partition is locally calibrated w.r.t a^{ABEE} if and only if (4) holds for $k = 1, \ldots, K-1$.

Straightforward computations show that condition (ii) is equivalent to $(a(\mu_k|\alpha_k) - \beta(\alpha_k))^2 \leq (a(\mu_k|\alpha_k) - \beta(\alpha_{k+1}))^2$, while condition (iii) is equivalent to $(a(\mu_k|\alpha_{k+1}) - \beta(\alpha_{k+1}))^2 \leq (a(\mu_k|\alpha_{k+1}) - \beta(\alpha_k))^2$. Hence conditions (i), (ii) and (iii) are necessary and sufficient conditions for an increasing sequence $\{\mu_k\}_{k=1}^K$ to generate An such that (a^{ABEE}, An) is a locally calibrated ABEE. **Q.E.D.**

Proposition 6 shows the range of interval partitions that can arise in locally calibrated ABEE. Condition (i) ensures that the sequence starts and ends at the extremes of the interval. Conditions (ii) and (iii) provide conditions required to build the sequence, given the first values μ_0 and μ_1 of the sequence. Since $\mathbb{E}[\mu|(\mu_k, \mu_{k+1}]]$ is increasing in μ_{k+1}, condition (ii) gives a lower bound on the set of μ_{k+1} that can be picked, given μ_{k-1} and μ_k. Similarly, condition (iii) gives an upper bound on μ_{k+1}, but this upper bound needs to be satisfied only if $\mathbb{E}[\mu|(\mu_{k-1}, \mu_k]] < \frac{1}{2C}$. Note that, when C tends to 0, the upper bound will always play a role. By contrast, when C tends to 1, the upper bound condition is not binding for μ_{k-1}, μ_k large enough.[26] Observe that when C tends to 0, the above conditions force the interval partition to be an equi-distant expectation partition. As C increases, more interval partitions can arise in a calibrated ABEE.

3.1.2 Globally calibrated ABEE

We now turn to the analysis of globally calibrated ABEE, and, to simplify the analysis, we focus on the case in which μ is uniformly distributed on $[0, 1]$. Since a globally calibrated ABEE must be a locally calibrated ABEE and the equal splitting interval partition gives rise to a locally calibrated ABEE in this case, we first explore when the stronger conditions for global calibration are satisfied for this interval partition.

[26] This is so because the slope of $a^{NE}(\mu)$ is constant for $C = 0$, and $a^{NE}(\mu)$ is more convex as C increases when $B > -AC$ (and concave for $B < -AC$). As we mentioned in the description of Figure 1, the size of the jumps is increasing with the steepness of $a^{NE}(\mu)$.

Proposition 7. *Let μ follow a uniform distribution over $[0,1]$. Denote by An the pair of symmetric analogy partitions generated by the equal splitting sequence and a^{ABEE} the corresponding ABEE. In the environment with strategic complements, there exist C^* and C^{**} such that i) for all $C < C^*$, (a^{ABEE}, An) is a globally calibrated ABEE; and ii) for all $C > C^{**}$ and $K > 3$, (a^{ABEE}, An) is not a globally calibrated ABEE.*

At some rough level, one might have thought that the first part of Proposition 7 derives from the observation that when C gets small, $a^{ABEE}(\mu)$ gets close to $A + B\mu$ for which the equal splitting partition is the only way to ensure global calibration. However, this intuition is incomplete, as we need to establish that for C away from 0 (even if small), the clustering of the datapoints $a^{ABEE}(\mu) = A + \mu \sum_{k=1}^{K} \mathbf{1}_{\{\mu_{k-1}<\mu\leq\mu_k]\}} \frac{B+AC}{1-C\mathbb{E}[\mu|\alpha_k]}$ leads to the equal splitting partition for global calibration purposes. In other words, we are not just requesting that for C small, the optimal clustering of $a^{ABEE}(\mu)$ be close to the equal splitting partition. We are requesting that it is exactly the equal splitting partition. This stronger requirement turns out to be satisfied because as we show that there is no partition other that the equal splitting partition that allows to satisfy the local conditions for local calibration with respect to $a^{ABEE}(\mu)$ when C is small enough.[27]

Regarding the second part of Proposition 7, we note that when C is above some threshold C^{**}, the function $a^{ABEE}(\mu)$ in the analogy class α_K (with highest μ) is so much steeper than in the first analogy class α_1 that it would be better -for variance minimization purposes- to split the last interval in half and merge the first two intervals together. Thus, for C large enough, the equal splitting partition does not satisfy the global optimality condition, and it cannot be part of a globally calibrated ABEE.

When the equal-splitting partition is not part of a globally calibrated ABEE (i.e. when C is too large), an open question is whether there exists another interval partition that can be used to support a globally calibrated ABEE. We would expect such an interval partition to use finer classes for larger values of μ so as to better reduce the variance in classes with large μ. While we can easily

[27] Moving away from the uniform distribution, we conjecture that the same result holds for a selection of equidistant-expectations partition (the one that solves the clustering problem when $C = 0$).

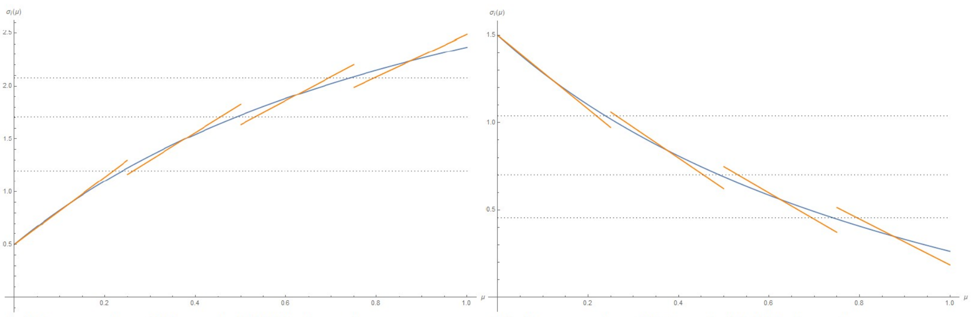
generate examples of globally calibrated ABEE with such features, we have not found a general argument showing the existence of a globally calibrated ABEE in all cases (as unlike for the analysis of locally calibrated ABEE we were not able to guess a potential candidate and verify whether the required conditions are satisfied).

3.2 Strategic Substitutes

We now assume that μ is distributed according to a continuous density function f with support $[-1, 0]$.

Note that, differently from the strategic complements environment, here the ABEE function $a^{ABEE}(\mu) = A + \sum_{k=1}^{K} \mathbf{1}_{\{\mu \in [\mu_{k-1}, \mu_k)\}} \mu \frac{B+AC}{1-C\mathbb{E}[\mu|\alpha_k]}$ is not monotone in μ. This is due to the fact that at the discontinuity points the jumps of the function are in the opposite direction with respect to the slope of $a(\mu|\alpha_k) = A + \mu \frac{B+AC}{1-C\mathbb{E}[\mu|\alpha_k]}$, and this is a fundamental difference induced by the change from strategic complements to strategic substitutes. To illustrate this, consider the case where $a(\mu|\alpha_k)$ has a positive slope, that is, $B < -AC$. Recall that $\beta(\alpha_k) = \frac{A+B\mathbb{E}[\mu|\alpha_k]}{1-C\mathbb{E}[\mu|\alpha_k]}$. Since μ is non-positive, $B < -AC$ and $\mathbb{E}[\mu|\alpha_k] < \mathbb{E}[\mu|\alpha_{k+1}]$ imply that $\beta(\alpha_k) < \beta(\alpha_{k+1})$. Since in the strategic substitutes environment the best-response is decreasing in the analogy-based expectations, at the adjacency point between two classes, the action played in equilibrium will be greater in the first of the two classes: $\beta(\alpha_k) < \beta(\alpha_{k+1})$ implies that $a(\mu_k|\alpha_k) > a(\mu_k|\alpha_{k+1})$. Hence, when $a(\mu|\alpha_k)$ is increasing in μ, the ABEE function jumps downwards at the discontinuity points. We can see this in the graphs of Figure 2.

Figure 2: Graphical Illustration of Local Calibration Requirements for $K = 4$ and μ-sequence such that $\mu_k = \frac{k}{4}$, $k = 0, 1, \ldots, 4$



(a) Increasing NE and ABEE functions
$A = 0.5$, $B = -4$ and $C = 0.9$

(b) Decreasing NE and ABEE functions
$A = 1.5$, $B = 1$ and $C = 0.9$

The non-monotonicities that arise in the ABEE strategies with interval analogy partitions in turn make it impossible to satisfy local calibration in a neighborhood of $\mu = \mu_k$. This is so because it cannot be simultaneously the case that $\lim_{\mu \to \mu_k^+} a^{ABEE}(\mu)$ is closer to $\beta((\mu_k, \mu_{k+1}])$ and $a^{ABEE}(\mu_k)$ is closer to $\beta((\mu_{k-1}, \mu_k])$.[28] Formally, we have:

Proposition 8. *In the strategic substitutes environment, whenever $B \neq -AC$, there are no symmetric interval analogy partitions that are locally calibrated with respect to the induced ABEE.*

Proposition 8 implies that there is no symmetric calibrated ABEE employing a single interval analogy partition. Given our general considerations in Section 2, it is natural to look for locally calibrated distributional ABEE and investigate whether we can have such equilibria with support of analogy partitions contained in the set of interval partitions. In our setup with a continuum of games, there are technical difficulties addressing this. In the online appendix, we consider a version with three values of μ, and we establish the existence of a symmetric calibrated distributional ABEE in this case.

4 Conclusion

In this paper we have introduced the notion of Calibrated ABEE defined so that i) given the analogy partitions, players choose strategies following the ABEE machinery, and ii) given the raw data on the opponent's strategies, players select analogy partitions following the K-means clustering prescriptions. We have observed that distributions over analogy partitions are sometimes required to guarantee existence whether local or global calibration is considered and whatever the notion of distance or divergence used at the clustering stage. We have applied our approach to one-dimensional families of games with linear best-responses, and shown that when games exhibit strategic complements, Calibrated ABEEs with symmetric interval partitions can arise, while mixing over partitions is needed for games with strategic substitutes. We hope our approach can fruitfully be applied in future

[28] For example when $B + AC > 0$, we would have $\beta((\mu_k, \mu_{k+1}]) > a^{ABEE}(\mu_k) > \lim_{\mu \to \mu_k^+} a^{ABEE}(\mu) > \beta((\mu_{k-1}, \mu_k])$, making it impossible to satisfy the local calibration conditions for $\mu = \mu_k$ and $\mu = \mu_k^+$ (i.e., μ slightly above μ_k).

works (theoretical, experimental or empirical) to shed more light on how economic agents categorize games into analogy partitions to form expectations about their opponents' strategies.

Appendix

The proofs of Proposition 4 and Lemmas 1, 2, 3 and 4 (which we have included for completeness) appear in the online Appendix.

Proof of Theorem 1. Compared to classic existence results in game theory, the main novelty is to show that the global calibration correspondence has properties that allow to apply Kakutani fixed point theorem to a grand mapping $M : \bar{\Sigma} \times \Lambda \rightrightarrows \bar{\Sigma} \times \Lambda$, which is a composition of the following functions and correspondences. Given $(\bar{\sigma}, \lambda)$ we compute the analogy-based expectations β through consistency and we call this function C. Given (β, λ), the Best Response correspondence (BR) yields the optimal strategies for each analogy partition in the support of λ. We aggregate the strategies following (2) obtaining $\bar{\sigma}'$ and define β' to be consistent with $\bar{\sigma}'$. We denote this function AG. We perform global calibration (GC) on $(\bar{\sigma}', \beta')$. Then, we obtain the following composition:

$$(\bar{\sigma}, \lambda) \mapsto_C (\beta, \lambda) \mapsto_{BR} (\sigma', \lambda) \mapsto_{AG} (\bar{\sigma}', \beta') \mapsto_{GC} (\bar{\sigma}', \lambda')$$

where $M(\bar{\sigma}, \lambda)$ denotes the set of $(\bar{\sigma}', \lambda')$ that can be obtained through this composition.

Note that C and AG are continuous functions, while BR and GC are correspondences. The mapping BR is upper-hemicontinuous (uhc) with non-empty, convex and compact values by standard arguments. Since, as we will prove later, GC is also upper-hemicontinuous (uhc) with non-empty, convex and compact values, it follows that:

(i) M is nonempty;

(ii) M is uhc as a composition of uhc mappings;

(iii) M is convex-valued since BR and GC are convex-valued;

(iv) M is compact-valued because BR being compact-valued and uhc implies that $BR(\beta, \lambda)$ is compact. Also, since, AG is single-valued and continuous, and GC is compact-valued and uhc, then $GC \circ AG \circ BR \circ C(\bar{\sigma}, \lambda)$ is compact, for all

$(\bar{\sigma}, \lambda) \in \bar{\Sigma} \times \Lambda$.

Since $\bar{\Sigma} \times \Lambda$ is a compact and convex set, properties (i) to (iv) ensure that M has a fixed point by Kakutani's theorem.

To conclude the proof we need to show the properties of the GC correspondence. GC maps $\bar{\Sigma} \times \bar{\Sigma}$ into $\bar{\Sigma} \times \Lambda$, where both $\bar{\Sigma}$ and Λ are convex and compact. The image of the correspondence is defined as follows:

$$GC(\bar{\sigma}, \beta) = \{\bar{\sigma}\} \cup \{\lambda \in \Lambda | \lambda_i(An_i) > 0 \iff An_i \in \arg\min_{An'_i \in \mathcal{K}_i} V(\bar{\sigma}_j, \beta'_i)\}$$

where $V_i(\bar{\sigma}_j, \beta'_i) = \sum_{\alpha_i \in An'_i} p(\alpha_i) \sum_{\omega \in \alpha_i} p(\omega|\alpha_i) d\left(\bar{\sigma}_j(\omega), \beta_i(\alpha_i|An'_i)\right)$.

For ease of exposition, let us denote the latter set in the union above as $G_i(\bar{\sigma}_j) \cup G_j(\bar{\sigma}_i)$. Note that G_i is nonempty because $\mathcal{K}_i$ is finite, thereby implying that there is always a solution to the minimization problem. Also, G_i is a simplex hence it is convex and compact. Thus, GC is nonempty, convex and compact valued. Let us check now that $GC(\bar{\sigma}, \beta)$ is upper-hemicontinuous, by verifying that it has a closed graph. In order to show this, we must prove that for $\bar{\sigma}^n_j \to \bar{\sigma}_j$ and $\lambda^n_i \to \lambda_i$, then $\lambda^n_i \in G_i(\bar{\sigma}^n_j) \Longrightarrow \lambda_i \in G_i(\bar{\sigma}_j)$.

We first establish the continuity of V_i by verifying that d is a continuous function. When d is the squared Euclidean distance, d is clearly continuous in $\bar{\sigma}$ and in β. Instead, when d represents the KL divergence, it is not generally continuous because whenever there is $\omega \in \alpha_i$ such that $supp[\bar{\sigma}_j(\omega)] \not\subset supp[\beta_i(\alpha_i|An_i)]$, then $d(\bar{\sigma}_j, \beta_i)$ goes to infinity. However, the consistency requirements impose $supp[\bar{\sigma}_j(\omega)] \subseteq supp[\beta_i(\alpha_i|An_i)]$. Since global calibration imposes for both players that β_i is consistent with $\bar{\sigma}$, for all ω, α_i and An_i, then $d(\bar{\sigma}_j, \beta_i)$ is finite. Recall that, $d(x, y) = \sum_a (x_a \ln x_a - x_a \ln y_a)$. Since $x_a, y_a \in [0, 1]$ and $x_a > 0$ implies $y_a > 0$, under the convention that $0 \ln 0 = 0$, d is continuous when it represents the KL divergence. Hence, V_i is continuous, if β_i is consistent with $\bar{\sigma}_j$ for both players.

We can now proceed to establish that GC is uhc. Assume by contradiction that $\lambda^n_i \to \lambda_i$ and $\lambda^n_i \in G_i(\bar{\sigma}^n_j)$, but $\lambda_i \notin G_i(\bar{\sigma}_j)$. Note that $\lambda_i \notin G_i(\bar{\sigma}_j)$ implies that there $\exists \tilde{An}_i \in \mathcal{K}_i | \lambda_i(\tilde{An}_i) > 0$ and $\varepsilon', \varepsilon > 0$ such that

$$+\infty > V_i(\bar{\sigma}_j, \tilde{\beta}_i) \geq V_i(\bar{\sigma}_j, \beta_i) + \varepsilon + \varepsilon',$$

where $\tilde{\beta}_i$ is consistent with $\bar{\sigma}_j$ according to $\tilde{An}_i$. Also, let $\tilde{\beta}^n_i$ be consistent with

$\bar{\sigma}_j^n$, according to $\tilde{An}_i$. We want to show that $\exists n \in \mathbb{N} | \lambda_i^n(\tilde{An}_i) > 0 \wedge V_i(\bar{\sigma}_j^n, \tilde{\beta}_i^n) > V_i(\bar{\sigma}_j^n, \beta_i^n)$. For $\lambda_i^n \to \lambda_i$ and $\lambda_i(\tilde{An}_i) > 0$, for any n large enough, $\lambda_i^n(\tilde{An}_i) > 0$. By continuity of V_i, when $\bar{\sigma}_j^n \to \bar{\sigma}_j$, for n large enough, $V_i(\bar{\sigma}_j, \beta_i) > V_i(\bar{\sigma}_j^n, \beta_i) - \varepsilon \geq V_i(\bar{\sigma}_j^n, \beta_i^n) - \varepsilon$, where the last inequality holds by Lemma 1. Then, $V_i(\bar{\sigma}_j, \tilde{\beta}_i) \geq V_i(\bar{\sigma}_j, \beta_i) + \varepsilon + \varepsilon' > V_i(\bar{\sigma}_j^n, \beta_i^n) + \varepsilon'$. Also, $\bar{\sigma}_j^n \to \bar{\sigma}_j$ implies that, for any n large enough, $V_i(\bar{\sigma}_j^n, \tilde{\beta}_i^n) > V_i(\bar{\sigma}_j, \tilde{\beta}_i^n) - \varepsilon' \geq V_i(\bar{\sigma}_j, \tilde{\beta}_i) - \varepsilon'$, where the last inequality holds by Lemma 1. Thus,

$$V_i(\bar{\sigma}_j^n, \tilde{\beta}_i^n) > V_i(\bar{\sigma}_j, \tilde{\beta}_i) - \varepsilon' \geq V_i(\bar{\sigma}_j, \beta_i) + \varepsilon > V_i(\bar{\sigma}_j^n, \beta_i^n)$$

We get $V_i(\bar{\sigma}_j^n, \tilde{\beta}_i^n) > V_i(\bar{\sigma}_j^n, \beta_i^n)$ and $\lambda_i^n(\tilde{An}_i) > 0$, which contradicts $\lambda_i^n \in G_i(\bar{\sigma}_j^n)$. It follows that GC is uhc. **Q.E.D.**

Proof of Proposition 2. The proof shares similarities with the purification techniques introduced by Harsanyi (1973). We consider the same grand mapping M that we introduced in the proof of Theorem 1, but now in the perturbed environment. The perturbations of the payoffs make best-responses single-valued as commonly observed in the previous learning literature. The main novelty here is that the perturbations on the strategies at the clustering stage make the calibration mapping single-valued too. The argument to show this is a bit more involved than for the payoff perturbation part because the perturbations at the clustering stage do not allow for additive separability.

More precisely, consider the compound mapping $(\bar{\sigma}, \lambda) \mapsto_C (\beta, \lambda) \mapsto_{BR} (\sigma', \lambda) \mapsto_{AG} (\bar{\sigma}', \beta')$ where $\bar{\sigma}'$ is the profile of aggregate best-responses, given λ.

Fix the probability distributions over analogy partitions λ. From the profile of aggregate strategies $\bar{\sigma} = (\bar{\sigma}_1, \bar{\sigma}_2)$ we can compute the corresponding analogy-based expectations $(\beta_i(\cdot|An_i))_{An_i \in supp\lambda_i}$ that are consistent with $\bar{\sigma}$. The mapping $(\bar{\sigma}, \lambda) \mapsto (\beta, \lambda)$ is continuous, single-valued and defined over convex and compact sets.

We consider best-responses in the perturbed environment. Let us order the actions in A_i, so that a_i^z is the z-th element in A_i. We denote by $a_i^*(\omega|An_i)(\cdot)$ the function that maps each realization of the profile of random variables $\tilde{\rho}_i$ for each action $a_i \in A_i$ in game ω into a best-response, and we write $a_i = a_i^*(\omega|An_i)(\rho_i)$ to indicate that a_i is played when the profile of realizations is $\rho_i(\omega) = (\rho_i(a_i', \omega))_{a_i'}$

where $\rho_i = (\rho_i(\omega))_\omega$. We denote by $X_i^z(a_i^*(\omega|An_i))$ the set of perturbations under which the action a_i^z is chosen according to $a_i^*(\omega|An_i)$. That is:

$$X_i^z(a_i^*(\omega|An_i)) = \{\rho_i | a_i^z = a_i^*(\omega|An_i)(\rho_i)\}. \tag{5}$$

The mixed strategy played by player i, under the analogy partition An_i in game ω is induced by $a_i^*(\omega|An_i)$ if and only if $\sigma_i(\omega|An_i)$ assigns probability $p_i(a_i^z;\omega,An_i)$ to a_i^z where

$$p_i(a_i^z;\omega,An_i) = \int \cdots \int_{\rho_i \in X_i^z(a_i^*(\omega|An_i))} d\rho_i(a_i^1,\omega)\ldots d\rho_i(a_i^{|A_i|},\omega) g_i(\rho_i(a_i^1,\omega))\ldots g_i(\rho_i(a_i^{|A_i|},\omega)) \tag{6}$$

and $g_i(\rho_i)$ is the continuously differentiable pdf of $\tilde{\rho}_i$.

Consider first the mapping $BR : (\beta,\lambda) \mapsto (\sigma',\lambda)$, where σ' is a profile of mixed strategies that is a best-response to β. Let $a_i^*(\omega|An_i)$ prescribe actions that are best responses to $\beta_i(\cdot|An_i)$, given the perturbed payoffs. Then BR is single-valued (because the set of realizations of the perturbations under which there are indifferences has measure zero), and it is readily verified that BR is a continuous function over convex and compact sets.

Consider the AG function, $(\sigma',\lambda) \mapsto (\bar{\sigma}',\beta')$ which aggregates the strategies over games and computes consistent expectations. AG is single-valued and continuous. Thus, the compound mapping $(\bar{\sigma},\lambda) \mapsto_C (\beta,\lambda) \mapsto_{BR} (\sigma',\lambda) \mapsto_{AG} (\bar{\sigma}',\beta')$ is also continuous and single valued over convex and compact sets.

For the calibration part, the argument is somewhat similar to the best response part. Consider the calibration mapping $GC : (\bar{\sigma}',\beta') \mapsto (\bar{\sigma}',\lambda')$, where $\lambda' = (\lambda_i', \lambda_j')$ is such that λ_i' solves the global clustering problem for player i.

Consider the perturbed strategies $\bar{s}$, where $\bar{s}_j(\omega) = \frac{\bar{\sigma}_j(\omega)+\varepsilon\eta_i(\omega)}{1+\varepsilon}$ and impose that β_i is consistent with $\bar{s}_j$. As established in Theorem 1, for each $\omega \in \alpha$, the function $d(\bar{s}_j(\omega), \beta_i(\alpha|An_i))$ is continuous in $\bar{s}_j$.

We define $An_i^*(\eta_i)$ as the function mapping the realization of the perturbation $\tilde{\eta}_i$ to an analogy partition An_i that solves the clustering problem. As before, we denote by $X_i^k(An_i^*) = \{\eta_i | An_i^k = An_i^*(\eta_i)\}$ the set of realizations such that the k-th analogy partition is prescribed by An_i^*.

The mixture of analogy partitions λ_i is induced by $An_i^*(\cdot)$ iff λ_i assigns probability $q(An_i^k)$ to An_i^k where $q(An_i^k) = \int \cdots \int_{\eta_i \in X_i^k(An_i^*)} d\eta_i(\omega_1)\ldots d\eta_i(\omega_N) h_i(\eta_i(\omega_1))....h_i(\eta_i(\omega_N))$, where h_i is the continuously differentiable pdf of η_i. We show now that the calibration mapping is single-valued. To establish this, we rely on results from chapter 2 in Milnor (1965). More precisely, we show that if An_i and An_i' yield the same V value (the criterion used for the clustering problem), then the set of realizations of $\tilde{\eta}_i$ that allow this has measure zero. Given, $\bar{\sigma}_j$ we can define the function $h(\eta_i) = V_i(\bar{s}_j, \beta_i(\cdot|An_i)) - V_i(\bar{s}_j, \beta_i(\cdot|An_i'))$, which is a mapping $h : U \to R$, where $\eta_i \in U$.[29] The function h is smooth (all partial derivatives exist and are continuous). Since $\eta_i(a_i, \omega) > 0$, for all ω and all a_i, then U is an open set. As $h(\eta_i) = 0$ is a regular value,[30] then the set $\{\hat{\eta}_i | h(\hat{\eta}_i) = 0\}$ is a smooth manifold of dimension $dim(U) - 1 = |A_i| \cdot |\Omega| - 1$, which has measure zero in U. Then, the argument for C being single valued and continuous are the same as those used for BR.

Thus, the compound mapping

$$M : (\bar{\sigma}, \lambda) \mapsto_C (\beta, \lambda) \mapsto_{BR} (\sigma', \lambda) \mapsto_{AG} (\bar{\sigma}', \beta') \mapsto_{GC} (\bar{\sigma}', \lambda')$$

is single-valued and continuous, and it maps $\bar{\Sigma} \times \Lambda$ into $\bar{\Sigma} \times \Lambda$, which are convex and compact sets. By Brouwer's fixed point theorem, this mapping has a fixed point.

It is then readily verified that the fixed point (σ, λ) is a steady state of the learning dynamics. **Q.E.D.**

Proof of Proposition 3. If $\lim_{\varepsilon \to 0}(\sigma^{(\varepsilon)}, \lambda^{(\varepsilon)}) = (\sigma, \lambda)$, then for ε small enough $supp[\sigma_i(\omega)] \subseteq supp[\sigma_i^{(\varepsilon)}(\omega)]$ and $supp[\lambda_i] \subseteq supp[\lambda_i^{(\varepsilon)}]$, for $i = 1, 2$ and $\omega \in \Omega$.

Since $(\sigma^{(\varepsilon)}, \lambda^{(\varepsilon)})$ is a steady state, any $An_i \in supp[\lambda_i^{(\varepsilon)}]$ solves the clustering problem for player i in the perturbed environment. Thus, for $\varepsilon = 0$, $An_i \in supp[\lambda_i]$ solves the clustering problem because d is continuous in ε (as established in Theorem 1, imposing consistency on β suffices to guarantee continuity in the case of KL divergence). The same argument can be made to show that σ is a best-response to λ. Thus, (σ, λ) is a steady state of the learning dynamics when $\varepsilon = 0$.

It follows that (σ, λ) is a globally calibrated ABEE because the requirements for the equilibrium and the steady states coincide when $\varepsilon = 0$ and the independence

[29] V_i is defined as in the proof of Theorem 1.

[30] To show this, we note that the first derivatives of $h(\cdot)$ wrt to $\eta_i(a_i, \omega)$ are linearly independent as one varies a_i and ω.

of the random draws ensures that $\sigma \in \Sigma_1 \times \Sigma_2$ and $\lambda \in \Lambda_1 \times \Lambda_2$. **Q.E.D.**

Proof of Propositions 5. We show Proposition 5 as a corollary of Proposition 6 (P6). Consider the increasing sequence $\{\mu_k\}_{k=0}^{K}$ with $\mu_0 = 0$, $\mu_K = 1$ and $\mu_k = \frac{\mathbb{E}[\mu|(\mu_{k-1},\mu_k]]+\mathbb{E}[\mu|(\mu_k,\mu_{k+1}]]}{2}$. The existence of such a sequence is ensured by Lemma 2. We simply check that the sequence we propose satisfies the conditions of P6.

Note that $\mu_0 = 0$ and $\mu_K = 1$, then condition (i) in P6 holds.

Condition (ii) in P6 requires that $\mathbb{E}[\mu|(\mu_k, \mu_{k+1}]] \geq \frac{2\mu_k - \mathbb{E}[\mu|(\mu_{k-1},\mu_k]]}{1+2C(\mu_k - \mathbb{E}[\mu|(\mu_{k-1},\mu_k]])}$. By substituting μ_k in the inequality with $\frac{\mathbb{E}[\mu|(\mu_{k-1},\mu_k]]+\mathbb{E}[\mu|(\mu_k,\mu_{k+1}]]}{2}$ we obtain:

$$\mathbb{E}[\mu|(\mu_k, \mu_{k+1}]] \geq \frac{\mathbb{E}[\mu|(\mu_k, \mu_{k+1}]]}{1 + 2C\mathbb{E}[\mu|(\mu_k, \mu_{k+1}]]}$$

which is true because the denominator is greater than 1.

To show that condition (iii) in P6 is satisfied, we need to check that, whenever $\mathbb{E}[\mu|(\mu_{k-1}, \mu_k]] < \frac{1}{2C}$, the following inequality holds: $\mathbb{E}[\mu|(\mu_k, \mu_{k+1}]] \leq \mu_k + \frac{\mu_k - \mathbb{E}[\mu|(\mu_{k-1},\mu_k]]}{1-2C\mathbb{E}[\mu|(\mu_{k-1},\mu_k]]}$. Recalling that $\mu_k = \frac{\mathbb{E}[\mu|(\mu_{k-1},\mu_k]]+\mathbb{E}[\mu|(\mu_k,\mu_{k+1}]]}{2}$, we obtain

$$\frac{\mathbb{E}[\mu|(\mu_k, \mu_{k+1}]] - \mathbb{E}[\mu|(\mu_{k-1}, \mu_k]]}{2} \leq \frac{\frac{\mathbb{E}[\mu|(\mu_k,\mu_{k+1}]]-\mathbb{E}[\mu|(\mu_{k-1},\mu_k]]}{2}}{1 - 2C\mathbb{E}[\mu|(\mu_{k-1}, \mu_k]]}$$

which holds because $0 < 1 - 2C\mathbb{E}[\mu|(\mu_{k-1}, \mu_k]] < 1$ when $\mathbb{E}[\mu|(\mu_{k-1}, \mu_k]] < \frac{1}{2C}$. **Q.E.D.**

Proof of Proposition 7. Let μ be uniformly distributed over $[0, 1]$. Let $\boldsymbol{\mu}^*$ denote the equal-splitting sequence where $\mu_k^* = \frac{k}{K}$, for $k = 0, \ldots, K$, and let $a^{ABEE}(\mu) = A + \mu \sum_{k=1}^{K} \mathbf{1}_{\{\mu_{k-1}^* < \mu \leq \mu_k^*\}} \frac{B+AC}{1-C\frac{\mu_{k-1}^*+\mu_k^*}{2}}$.

C small

We want to show that, for C small enough, the equal splitting μ^* generates the unique symmetric analogy partition profile that is locally calibrated wrt a^{ABEE}. We proceed in the following way: we approximate a^{ABEE} with Taylor expansions (in C) around $C = 0$, and we establish a necessary condition for local calibration wrt the induced approximated strategies. Then, starting nearby (but away from) μ^* for low values of μ_k, we build the next μ_k in the sequence so that the local calibration requirements are satisfied for those. We show that the final value in the sequence μ_K cannot be equal to 1, thereby establishing a contradiction to the existence of a locally calibrated sequence other than μ^*.

More precisely, for $C = 0$, the upper bound and lower bound identified in Proposition 6 coincide and it is readily verified that the unique locally calibrated -thus, globally calibrated- partition wrt a^{ABEE} is generated by the equal-splitting sequence. For C small enough, any solution to the minimization problem is in a neighborhood of μ^*.

Consider first the case where both B and C are close to zero, the 1st order Taylor expansion around $(B, C) = (0, 0)$ yields $a^{ABEE}(\mu) \approx A + \mu(B + AC)$, for $\mu \in [0, 1]$. This is a continuous line so the equal-splitting would satisfy global calibration. Hence, we can now restrict attention to situations where B is not close to zero. First-order approximation around $C = 0$ yields:

$$a^{ABEE}(\mu) \approx A + \mu(S + \sum_{k=1}^{K} \mathbf{1}_{\{\mu^*_{k-1} < \mu \leq \mu^*_k\}}(k-1)D) \equiv a(\mu)$$

where $S = B + C(A + \frac{B}{2K})$ and $D = \frac{BC}{K}$. Since B is not close to zero, for C small enough, $a(\mu)$ is strictly increasing (decreasing) in μ for $B > 0(<)$ and its slope is weakly increasing (decreasing) in μ. We let β be the expectations that are consistent with the approximated action function $a(\mu)$. We focus on the case with $B > 0$ (the argument for $B < 0$ is analogous).

We establish that any increasing sequence $\{\mu_\ell\}_{\ell=0}^K$ generating analogy partitions that are locally calibrated wrt $a(\mu)$ must satisfy the condition that $\mu_{\ell+1} \leq 2\mu_\ell - \mu_{\ell-1}$, for all ℓ such that $a(\mu)$ is continuous at $\mu = \mu_\ell$: whenever $\mu^*_k < \mu_\ell < \mu^*_{k+1}$, local calibration requirements for μ_ℓ reduce to $\beta(\mu_\ell, \mu_{\ell+1}) = 2a(\mu_\ell) - \beta(\mu_{\ell-1}, \mu_\ell)$. If $a(\mu)$ were a straight line, this condition would imply that $\mu_{\ell+1} = 2\mu_\ell + \mu_{\ell-1}$. But $a(\mu)$ has discontinuities and its slope is weakly increasing in μ, so it must be the case that $\mu_{\ell+1} \leq 2\mu_\ell + \mu_{\ell-1}$.

We now consider sequences in a neighborhood of μ^*. We distinguish between two cases: we show that any sequence that starts below or above μ^* cannot satisfy locally calibration requirements and end at $\mu_K = 1$.

Case 1.

Let $\{\mu_k\}_{k=0}^K$ be an increasing sequence with $\mu_{k'} = \mu^*_{k'}$, for all $0 \leq k' < k-1 < K$ and $\mu_{k-1} < \mu^*_{k-1}$. Assume that $\{\mu_k\}_{k=0}^K$ satisfies the conditions for local calibration wrt to $a(\mu)$. As established above, local calibration requires $\mu_k \leq 2\mu_{k-1} - \mu_{k-2} < 2\mu^*_{k-1} - \mu^*_{k-2} = \mu^*_k$. That is, the k-th element in the sequence will also be below the respective element in μ^*. And not only this: it must also be the case that the interval from μ_{k-1} to μ_k is shorter than $1/K$, the length of intervals for μ^*. To see this, note that $\mu_k - \mu_{k-1} \le \mu_{k-1} - \mu_{k-2} < \mu^*_{k-1} - \mu^*_{k-2} = \frac{1}{K}$. We show by induction that the sequence must be such that $\mu_K < \mu^*_K = 1$. Assume the following induction hypothesis, for some integer $k \le \ell < K$:

$$(hp): \begin{cases} \mu_{\ell-2} \le \mu^*_{\ell-2} \\ \mu_{\ell-1} < \mu^*_{\ell-1} \\ \mu_{\ell-1} - \mu_{\ell-2} < \mu^*_{\ell-1} - \mu^*_{\ell-2} \end{cases} \implies \begin{cases} \mu_\ell < \mu^*_\ell \\ \mu_\ell - \mu_{\ell-1} < \mu^*_\ell - \mu^*_{\ell-1} \end{cases}$$

we show that hp implies that, under local calibration, $\mu_{\ell+1} < \mu^*_{\ell+1} \wedge \mu_{\ell+1} - \mu_\ell < \mu^*_{\ell+1} - \mu^*_\ell$.

If $\nexists t \in \{k, \dots, \ell-1\}$ s.t. $\mu_\ell = \mu^*_t$, then $a(\mu)$ is continuous at μ_ℓ and we have already proven that local calibration requires $\mu_{\ell+1} \le 2\mu_\ell - \mu_{\ell-1}$. By hp, we assume that $2\mu_\ell - \mu_{\ell-1} < 2\mu^*_\ell - \mu^*_{\ell-1} = \mu^*_{\ell+1}$, so that $\mu_{\ell+1} < \mu^*_{\ell+1}$. Also, $\mu_{\ell+1} - \mu_\ell \le \mu_\ell - \mu_{\ell-1} < \mu^*_\ell - \mu^*_{\ell-1} = \mu^*_{\ell+1} - \mu^*_\ell$.

If instead $\mu_\ell = \mu^*_t$, for some $t \le \ell - 1$, the function $a(\mu)$ is discontinuous at μ_ℓ and, when $B > 0$, local calibration requires $2a(\mu_\ell^-) \le \beta(\mu_{\ell-1}, \mu_\ell) + \beta(\mu_\ell, \mu_{\ell+1}) \le 2a(\mu_\ell^+)$, where $a(\mu_\ell^-) = A + \mu^*_t(S + (t-1)D)$ and $a(\mu_\ell^+) = A + \mu^*_t(S + tD)$.

By hp, it must be the case that $\mu^*_{t-1} < \mu_{\ell-1} < \mu^*_t$, so we write $\mu_{\ell-1} = \mu^*_t - \varepsilon$, for $0 < \varepsilon < \frac{1}{K}$ and $\beta(\mu_{\ell-1}, \mu_\ell) = A + (\mu^*_t - \frac{\varepsilon}{2})(S + (t-1)D))$. Assume that $\mu_{\ell+1} = \mu^*_{t+1}$. Then $\beta(\mu_\ell, \mu_{\ell+1}) = A + \frac{\mu^*_t + \mu^*_{t+1}}{2}(S + tD)$. Therefore, the local calibration condition for $B > 0$ writes (after rearranging)

$$0 \le (\mu^*_{t+1} - \mu^*_t - \varepsilon)(S + (t-1)D) + (\mu^*_t + \mu^*_{t+1})D \le 4\mu^*_t D$$

as $C \to 0$, $D \to 0$, while $S \to B$, at the limit we would obtain $0 \le (\mu^*_{t+1} - \mu^*_t - \varepsilon)B \le 0$. But then, there exists C small enough such that $(\mu^*_{t+1} - \mu^*_t - \varepsilon)(S + (t-1)D) + (\mu^*_t + \mu^*_{t+1})D > 4\mu^*_t D$, which implies that, when $\mu_{\ell+1} = \mu^*_{t+1}$, then $\beta(\mu_\ell, \mu_{\ell+1})$ is too large to satisfy the local calibration conditions. Then, the induction argument works even when we hit discontinuity points along the sequence.

Noticing that the induction hypothesis holds for $\ell = k$, then we have $\mu_K < 1$. That is, for C small enough, any sequence that goes below μ^* at some point cannot be locally calibrated and end at 1 at the same time.

Case 2. (i.e. when a sequence goes above μ^* at some point) is treated analogously. See details in the online appendix.

Hence, for C small enough, the only locally calibrated sequence in a neighborhood of μ^* is μ^*. **Q.E.D.**

C large

We now show that, for $K \geq 4$, when C is large enough, the analogy partition An^* generated by μ^* is not a globally calibrated wrt a^{ABEE}.

We establish this showing that for C large enough less variance is induced by the alternative analogy partition An', generated by the sequence $\{\mu'_k\}_{k=0}^K$, where the first two intervals of the equal splitting sequence are merged together and the last interval is split in half. That is, $\mu'_0 = 0$, $\mu'_K = 1$, $\mu'_{K-1} = \frac{\mu^*_{K-1}+\mu^*_K}{2} = \frac{2K-1}{2K}$ and $\mu'_k = \mu^*_{k+1} = \frac{k+1}{k}$, for $k = 1, \ldots, K-2$. We compare the variance induced by An^* and An', denoted respectively $Var^*(An^*)$ and $Var^*(An')$ when a^{ABEE} is generated by the equal splitting partition.

Under μ^*, the within variance of the analogy class α^*_k is

$$Var(\alpha^*_k) = \int_{\mu^*_{k-1}}^{\mu^*_k} \frac{(a^{ABEE}(\mu) - \beta(\alpha^*_k))^2}{\mu^*_k - \mu^*_{k-1}} d\mu = \left(\frac{B + AC}{1 - C\frac{\mu^*_k+\mu^*_{k-1}}{2}} \right)^2 \frac{(\mu^*_k - \mu^*_{k-1})^2}{12}$$

and $Var^*(An^*) = \sum_{k=1}^K (\mu^*_k - \mu^*_{k-1}) Var(\alpha^*_k)$.

Note that by construction, $Var(\alpha'_k) = Var(\alpha^*_{k+1})$, for $k = 2, \ldots, K-2$. We need to compute the variance for α'_1, α'_{K-1} and α'_K. Since α'_1 consists of the first two merged classes, $\beta(\alpha'_1) = \frac{\beta(\alpha^*_1)+\beta(\alpha^*_2)}{2}$, while $\beta(\alpha'_{K-1}) = \frac{A(2-C(\mu'_K-\mu'_{K-1})+B(\mu'_{K-2}+\mu'_{K-1})}{2-C(\mu'_K+\mu'_{K-2})}$ and

$\beta(\alpha'_K) = \frac{A(2-C(\mu'_{K-2}-\mu'_{K-1})+B(\mu'_K+\mu'_{K-1})}{2-C(\mu'_K+\mu'_{K-2})}$. We can now proceed to compute the variance in the analogy classes (algebra is omitted). Recalling that $\mu'_1 = 2\mu^*_1$, then direct computation shows

$$Var(\alpha'_1) = \frac{1}{2\mu^*_1} \int_0^{2\mu^*_1} (a^{ABEE}(\mu) - \beta(\alpha'_1))^2 d\mu = \mu^{*2}_1 \frac{(B + AC)^2(5C^2\mu^{*2}_1 - 8C\mu^*_1 + 16)}{3(2 - 3C\mu^*_1)^2(2 - C\mu^*_1)^2}$$

Also, from the fact the last two analogy classes of An' are along the same segment and have the same length:

$$Var(\alpha'_{K-1}) = Var(\alpha'_K) = \frac{(B+AC)^2}{(1-C^{\frac{2K-1}{2}}\mu_1^*)^2}\frac{\mu_1^{*2}}{48}$$

Recalling that $\mu_1^* = \frac{1}{K}$, we get that if $K \geq 4$, then

$$\lim_{C\to 1} Var^*(An^*) - Var^*(An') = \frac{(B+A)^2(8K(2K^3-8K^2+7K-6)+9)}{4K(4K^2-8K+3)^2} > 0\,.$$

Q.E.D.

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Online Appendix

Proof of Lemma 1.

Consider first the case of squared Euclidean distance.

Let d be the squared Euclidean distance and define a loss function as

$L = \sum_{\omega \in \alpha_i} p(\omega|\alpha i) \sum_{a \in A_j} (\sigma_{j,a}(\omega) - q_a)^2$, where $\sigma_{j,a}(\omega)$ indicates the probability that action a is played according to strategy $\sigma_j(\omega)$. Then, the FOCs are

$$q_a^* = \sum_{\omega \in \alpha_i} p(\omega|\alpha_i)\sigma_{j,a}(\omega),\ \forall a \in A_j$$

Moreover, $\frac{\partial^2 L}{\partial q_a^2} = 2 > 0$: L is strictly convex and q_a^* minimizes the function on the interior of the simplex.

Notice that the equations obtained for q^* allow for $q_a^* = 0$ whenever a is not in the support of σ_j. In some sense, conditions in q^* include also some corner solutions. Can corner solutions different from q^* be a global minimum? If $\sigma_{j,a}(\omega) = 0, \forall \omega \in \alpha_i$, then $q_a = 0$ would allow to minimize the objective function, and the equations above can describe this instance. Assume that $\exists (\hat{a}, q) \in A_j \times \Delta A_j \text{s.t.} q_{\hat{a}} = 0 \text{and} \hat{a} \in Supp\sigma_j$. Since $\sum_a q_a = 1$ and $q_{\hat{a}}^* > 0$, by FOCs, then $q_{\hat{a}} = 0 < q_{\hat{a}}^*$, which implies that $\exists a\prime \in A_j | q_{a\prime} > q_{a\prime}^*$. Let q^ε be such that $q_a^\varepsilon = q_a, \forall a \in A_j \setminus \{\hat{a}, a\prime\}$, and let $q_{\hat{a}}^\varepsilon = \varepsilon$ and $q_{a\prime}^\varepsilon = q_{a\prime} - \varepsilon$, where $\varepsilon \in (0, q_{a\prime} - q_{a\prime}^*]$. For $q_{a\prime}^* \leq q_{a\prime}^\varepsilon < q_{a\prime}$, $\sum_{\omega \in \alpha_i} p(\omega|\alpha_i)(\sigma_{j,a\prime}(\omega) - q_{a\prime})^2 > \sum_{\omega \in \alpha_i} p(\omega|\alpha_i)(\sigma_{j,a\prime}(\omega) - q_{a\prime}^\varepsilon)^2$ by FOCs and strict convexity. Also, $\sum_{\omega \in \alpha_i} p(\omega|\alpha_i)(\sigma_{j,\hat{a}}(\omega) - q_{\hat{a}})^2 - \sum_{\omega \in \alpha_i} p(\omega|\alpha_i)(\sigma_{j,\hat{a}}(\omega) - q_{\hat{a}}^\varepsilon)^2 = \varepsilon(2\sum_{\omega \in \alpha_i} p(\omega|\alpha_i)\sigma_{j,\hat{a}}(\omega) - \varepsilon) > 0$ if and only if $0 < \varepsilon < 2q_{\hat{a}}^*$.
Then, for $\bar{\varepsilon} \in \{\varepsilon \in (0, q_{a\prime} - q_{a\prime}^*] | \varepsilon < 2q_{\hat{a}}^*\}$, $q^{\bar{\varepsilon}}$ improves upon the objective compared to q. Hence, q cannot be the global minimum. Then, q^* is the global minimum of L.

Consider the case of Kullback-Leibler divergence.

Let d be the Kullback-Leibler divergence and define the loss function as

$L = \sum_{\omega \in \alpha_i} p(\omega|\alpha_i) \sum_{a \in A_j} \sigma_{j,a}(\omega) \ln \frac{\sigma_{j,a}(\omega)}{q_a}$. Note that, if some q minimizes $L+1$, then it minimizes L. Recall that q is a probability distribution, so $\sum_{a \in A_j} q_a = 1$. Therefore we minimize the function

$L+1 = \sum_{\omega\in\alpha_i} p(\omega|\alpha_i) \sum_{a\in A_j} \sigma_{j,a}(\omega) \ln \frac{\sigma_{j,a}(\omega)}{q_a} + \sum_{a\in A_j} q_a$. The FOCs are:

$$q_a^* = \sum_{\omega\in\alpha_i} p(\omega|\alpha_i)\sigma_{j,a}(\omega),\ \forall a \in A_j$$

Moreover, $\frac{\partial^2}{\partial q_a^2}(L+1) = 1 > 0$: $L+1$ is strictly convex and q_a^* minimizes the function.
Analogously to the case of Euclidean distance, also in this case the conditions for q^* yield a global minimum. In particular, if $\exists(\hat{a}, q) \in A_j \times \Delta A_j \text{s.t.} q_{\hat{a}} = 0 \text{and} \hat{a} \in Supp\sigma_j$, then, $d_{KL}(\sigma_j(\omega), q) \to \infty$. Thus, q^* is a global minimum of L. **Q.E.D.**

Lemma 3. *Let σ be some strategy profile and d be either the squared Euclidean distance or the KL divergence. If $An_i \in \mathcal{K}_i$ is a globally calibrated analogy partition for player i with respect to σ, then An_i is a locally calibrated analogy partition for player i with respect to σ.*

Proof. Let An_i be a globally calibrated analogy partition with respect to σ. Let β_i be consistent with σ. Assume by contradiction that An_i is not locally calibrated. Then, $\exists \alpha_i, \alpha\prime_i \in An_i \wedge \hat{\omega} \in \alpha_i$ s.t. $d(\sigma_j(\hat{\omega}), \beta_i(\alpha_i)) > d(\sigma_j(\hat{\omega}), \beta_i(\alpha\prime_i))$.
Let $\widehat{\alpha_i} = \alpha_i \setminus \{\hat{\omega}\}$ and $\widehat{\alpha\prime_i} = \alpha\prime_i \cup \{\hat{\omega}\}$. Then,

$$\sum_{\omega\in\alpha_i} p(\omega)d(\sigma_j(\omega), \beta_i(\alpha_i)) + \sum_{\omega\in\alpha\prime_i} p(\omega)d(\sigma_j(\omega), \beta_i(\alpha\prime_i))$$

$$> \sum_{\omega\in\widehat{\alpha_i}} p(\omega)d(\sigma_j(\omega), \beta_i(\alpha_i)) + \sum_{\omega\in\widehat{\alpha\prime_i}} p(\omega)d(\sigma_j(\omega), \beta_i(\alpha\prime_i))$$

$$= p(\widehat{\alpha_i}) \sum_{\omega\in\widehat{\alpha_i}} p(\omega|\widehat{\alpha_i})d(\sigma_j(\omega), \beta_i(\alpha_i)) + p(\widehat{\alpha\prime_i}) \sum_{\omega\in\widehat{\alpha\prime_i}} p(\omega|\widehat{\alpha\prime_i})d(\sigma_j(\omega), \beta_i(\alpha\prime_i))$$

$$> p(\widehat{\alpha_i}) \sum_{\omega\in\widehat{\alpha_i}} p(\omega|\widehat{\alpha_i})d(\sigma_j(\omega), \beta_i(\widehat{\alpha_i})) + p(\widehat{\alpha\prime_i}) \sum_{\omega\in\widehat{\alpha\prime_i}} p(\omega|\widehat{\alpha\prime_i})d(\sigma_j(\omega), \beta_i(\widehat{\alpha\prime_i}))$$

where the second inequality holds by Lemma 1.
Let $\widehat{An_i} = \widehat{\alpha_i} \cup \widehat{\alpha\prime_i} \cup \{An_i \setminus \{\alpha_i, \alpha\prime_i\}\}$, then:

$$\sum_{\alpha_i\in An_i} p(\alpha_i) \sum_{\omega\in\alpha_i} p(\omega|\alpha)d(\sigma_j(\omega), \beta_i(\alpha_i)) > \sum_{\alpha_i\in\widehat{An_i}} p(\alpha_i) \sum_{\omega\in\alpha_i} p(\omega|\alpha_i)d(\sigma_j(\omega), \beta_i(\alpha_i))$$

which contradicts An_i being a globally calibrated analogy partition. **Q.E.D.**

Proof of Proposition 4.

By definition, $\sigma_j(\mu) \in BR_j(\mu, \beta_i(\alpha_k))$ implies that the following equations must hold in equilibrium:

$$\sigma_j(\mu) = A + \mu B + \mu C \int_{\mu_{k-1}}^{\mu_k} \frac{f(\mu)}{F(\mu_k) - F(\mu_{k-1})} \sigma_i(\mu) d\mu$$

$$= A + \mu \left(B + AC + BC\mathbb{E}[\mu|\alpha_k] + C^2\mathbb{E}[\mu|\alpha_k] \int_{\mu_{k-1}}^{\mu_k} \frac{f(\nu)}{F(\mu_k) - F(\mu_{k-1})} \sigma_j(\nu) d\nu \right)$$

Taking the weighted average of $\sigma_j(\mu)$ over the interval $[\mu_{k-1}, \mu_k]$, according to the distribution of μ, yields the following equation:

$$\int_{\mu_{k-1}}^{\mu_k} \frac{f(\mu)}{F(\mu_k) - F(\mu_{k-1})} \sigma_j(\mu) d\mu$$

$$= A + \mathbb{E}[\mu|\alpha_k] \left(B + AC + BC\mathbb{E}[\mu|\alpha_k] + C^2\mathbb{E}[\mu|\alpha_k] \int_{\mu_{k-1}}^{\mu_k} \frac{f(\nu)}{F(\mu_k) - F(\mu_{k-1})} \sigma_j(\nu) d\nu \right)$$

By consistency of $\beta_i(\alpha_k)$, the equation above simplifies into $\beta_i(\alpha_k) = \frac{A + B\mathbb{E}[\mu|\alpha_k]}{1 - C\mathbb{E}[\mu|\alpha_k]}$.

In equilibrium, both players have the same expectations $\beta_1(\alpha_k) = \beta_2(\alpha_k)$. Substituting the expression of $\beta_i(\alpha_k)$ into the best-responses yields the following equilibrium (pure) strategies: for all $\alpha_k \in An_1$ (and An_2) and for all $\mu \in \alpha_k$,

$$\sigma_1(\mu) = \sigma_2(\mu) = A + \mu \frac{B + AC}{1 - C\mathbb{E}[\mu|\alpha_k]}$$

This is the unique ABEE and it is symmetric. **Q.E.D.**

Lemma 4. *(Reverse Truncation) Let X be a RV on $[0,1]$ with continuous pdf f_X and cdf F_X. There exists a continuous pdf g over $[0, +\infty)$, such that:*

$$g_{X|[0,1]}(x) = f_X(x), \textit{ where } g_{X|[0,1]}(x) = \frac{g_X(x)}{Pr_g[0 \leq X \leq 1]}$$

Proof of Lemma 4.

We want to find a continuous function g_X such that:

$$g_X(x) = \begin{cases} Pr_g[0 \leq X \leq 1] f_X(x) & 0 \leq x \leq 1 \\ v(x) & 1 < x < \infty \end{cases}$$

Note that g_X is continuous if $v(\cdot)$ is continuous and $v(1) = Pr_g[0 \le X \le 1] f_X(1)$. Also, g_X must be a pdf, so it must be the case that $\int_0^{+\infty} g_X(x)dx = 1$. That is, $\int_0^1 Pr_g[0 \le X \le 1] f_X(x)dx + \int_1^{+\infty} v(x)dx = Pr_g[0 \le X \le 1] + (1 - Pr_g[0 \le X \le 1]) = 1$.
Let us pick the right function $v(.)$. This function must be continuous and satisfy two conditions: (i) $\int_1^{+\infty} v(x)dx = 1 - T$, and (ii) $v(1) = T f_X(1)$, where $T \equiv Pr_g[0 \le X \le 1]$. Let $v()$ be defined as $v(x) = Tf(1)e^{\frac{Tf(1)}{1-T}(1-x)}$ which is continuous since, for $a, b \in \mathbb{R}$, the function ae^{bx} is continuous in x. Also, $v(1) = Tf(1)e^{\frac{Tf(1)}{1-T}(0)} = Tf(1)$.

And finally: $\int_1^{+\infty} v(x)dx = Tf(1)e^{\frac{Tf(1)}{1-T}} \int_1^{+\infty} e^{-\frac{Tf(1)}{1-T}x}dx = 1 - T$. **Q.E.D.**

Proof of Lemma 2.

To prove the existence result in Lemma 2, we show that there always exists a sequence $\{\mu_k\}_{k=0}^{k}$ with $\mu_0 = 0$, some $\mu_1 \in [0, 1]$ and, for $k = 2, \ldots, K$, μ_k defined so that $\mathbb{E}[\mu|(\mu_{k-1}, \mu_k]] = 2\mu_{k-1} - \mathbb{E}[\mu|(\mu_{k-2}, \mu_{k-1}]]$. We note that if μ_1 is too large, μ_K might be above 1. Lemma 4 allow us to consider μ to be a random variable from $[0, +\infty)$ distributed according to a continuous strictly positive pdf g, and cdf G, with $g(\mu) = f(\mu)G(1)$, for $0 \le \mu \le 1$ (see the online appendix for details).

Let $\mu_k \ge \mu_{k-1} \ge 0$. Since $g(\mu)$ is stricly positive and continuous in μ, then $\mathbb{E}[\mu|(\mu_{k-1}, \mu_k]] = \frac{1}{G(\mu_k) - G(\mu_{k-1})} \int_{\mu_{k-1}}^{\mu_k} \mu g(\mu)d\mu$ is continuous and strictly decreasing in μ_{k-1}, and it is continuous and strictly increasing in μ_k. Moreover, if $\mu_k \le 1$, we have:

$$\frac{\int_{\mu_{k-1}}^{\mu_k} \mu g(\mu)d\mu}{G(\mu_k) - G(\mu_{k-1})} = \frac{\int_{\mu_{k-1}}^{\mu_k} \mu G(1)f(\mu)d\mu}{G(1)(F(\mu_k) - F(\mu_{k-1}))}$$

so the term $G(1)$ cancels out and we are back to the original distribution F.

Fix μ_{k-1}. The function $m(\mu_k) \equiv \mathbb{E}[\mu|[\mu_{k-1}, \mu_k]]$ is continuous and strictly increasing over $(\mu_{k-1}, +\infty)$, with image $(\mu_{k-1}, +\infty)$. Then the inverse function m^{-1} exists over $(\mu_{k-1}, +\infty)$ and it is continuous and strictly increasing over $(\mu_{k-1}, +\infty)$.

Given μ_{k-2}, μ_{k-1}, we use the inverse function to retrieve μ_k from the equation $\mathbb{E}[\mu|(\mu_{k-1}, \mu_k]] = 2\mu_{k-1} - \mathbb{E}[\mu|(\mu_{k-2}, \mu_{k-1}]]$. Let $h(\mu_{k-2}, \mu_{k-1}) \equiv 2\mu_{k-1} - \mathbb{E}[\mu|(\mu_{k-2}, \mu_{k-1}]]$. Note that $h(\cdot)$ is a continuous function and $h(\mu_{k-2}, \mu_{k-1}) \ge \mu_{k-1}$.

Starting from $\mu_0 = 0$ and some $\mu_1(\mu_1) \equiv \mu_1$, we recursively define μ_k as a function of μ_1 as follows: $\mu_2(\mu_1) = m^{-1}(h(\mu_0, \mu_1))$ and

$$\mu_k(\mu_1) = m^{-1}(h(\mu_{k-2}(\mu_1), \mu_{k-1}(\mu_1)))$$

for $k = 3, \dots, K$. Note that, for each k, the function $\mu_k(\mu_1)$ is well defined. Since $h(\mu_{k-2}, \mu_{k-1}) \geq \mu_{k-1}$, then the inverse function exists at the point $h(\mu_{k-2}, \mu_{k-1})$.

Since $m^{-1} : (\mu_{k-1}, +\infty) \to (\mu_{k-1}, +\infty)$ is also strictly increasing and continuous, then $\mu_k(\mu_1)$ is continuous, being a composition of continuous functions, and $\mu_k(\mu_1) \geq \mu_{k-1}$, with equality if and only if $h(\mu_{k-2}, \mu_{k-1}) = \mu_{k-1} \iff \mu_{k-1} = \mathbb{E}[\mu|(\mu_{k-2}, \mu_{k-1}]] \iff \mu_{k-1} = \mu_{k-2}$. So, either we get the sequence with 0 everywhere, or a strictly increasing sequence.

Let $\mu_1 = 0$, then $\mu_K(0) = 0$. Let $\mu_1 = 1$, then $\mu_K(1) > 1$. Then, by the intermediate value theorem, there must exist $0 < \mu_1^* < 1$ such that $\mu_K = 1$. **Q.E.D.**

Case 2 in Proof of Propostion 7

Let $\{\mu_k\}_{k=0}^K$ be an increasing sequence with $\mu_{k'} = \mu^*_{k'}$, for all $0 \leq k' < k-1 < K$ and $\mu^*_{k-1} < \mu_{k-1}$. Assume that $\{\mu_k\}_{k=0}^K$ satisfies the conditions for local calibration wrt to $a(\mu)$. Let $\mu_{k-1} = \mu^*_{k-1} + \varepsilon$, where $0 < \varepsilon < \frac{1}{2^K K}$. Note that this assumption implies that the $a(\mu)$ is continuous at μ_{k-1} and we have shown that, for local calibration, it must be the case that $\mu_k \leq 2\mu_{k-1} - \mu_{k-2} = \mu^*_k + 2\varepsilon$. We show that the local calibration requirements imply that the next element in the sequence should be greater than the respective element in μ^* and that the interval should be larger than $\frac{1}{K}$, i.e. $\mu_k > \mu^*_k$ and $\mu_k - \mu_{k-1} > \mu^*_k - \mu^*_{k-1}$.

Since $a(\mu)$ is continuous at μ_{k-1}, local calibration requires that $2a(\mu_{k-1}) = \beta(\mu_{k-2}, \mu_{k-1}) + \beta(\mu_{k-1}, \mu_k)$. We have that $a(\mu_{k-1}) = A + \mu_{k-1}(S + kD)$ and

$$\beta(\mu_{k-2}, \mu_{k-1}) = \int_{\mu_{k-2}}^{\mu_{k-1}} a(\mu) f(\mu|\mu_{k-2} \leq \mu \leq \mu_{k-1}) d\mu = A + \frac{\mu_{k-2} + \mu_{k-1}}{2}(S + (k-1)D) + \frac{\mu^2_{k-1} - \mu^{*2}_{k-1}}{2(\mu_{k-1} - \mu_{k-2})} D$$

where the last term of the expression comes from the fact that there is a jump at μ^*_{k-1}. Let $\mu_k = \mu^*_k + \varepsilon$. Similarly, $\beta(\mu_{k-1}, \mu_k) = A + \frac{\mu_{k-1}+\mu_k}{2}(S + kD) + \frac{\mu_k^2 - \mu_k^{*2}}{2(\mu_k - \mu_{k-1})} D$. For $B > 0$, we check that the chosen μ_k is too small, when C is small enough:

$$\beta(\mu_{k-1}, \mu_k) < 2a(\mu_{k-1}) - \beta(\mu_{k-2}, \mu_{k-1}) \iff$$

$$(\mu_k + \mu_{k-1})(S + kD) + \frac{\mu_k^2 - \mu_k^{*2}}{\mu_k - \mu_{k-1}} D < (3\mu_{k-1} - \mu_{k-2})(S + kD) + \frac{\mu^{*2}_{k-1} - \mu^2_{k-2}}{\mu_{k-1} - \mu_{k-2}} D \iff$$

$$\frac{\mu_k^2-\mu_k^{*2}}{\mu_k-\mu_{k-1}}D<\varepsilon(S+kD)+\frac{\mu_{k-1}^{*2}-\mu_{k-2}^2}{\mu_{k-1}-\mu_{k-2}}D$$

where the last equivalence hold because $\mu_{k-2}=\mu_{k-2}^*$, $\mu_{k-1}=\mu_{k-1}^*+\varepsilon$ and $\mu_k=\mu_k^*+\varepsilon=2\mu_{k-1}^*-\mu_{k-2}^*$. As $C\to 0$, the last inequality is satisfied because $D\to 0$ and $S\to B>0$.

We now proceed to show by induction that this sequence must be such that $\mu_K>1$. Assume the following induction hypothesis (hp'), for some integer $k\le \ell<K$:

$$(hp'):\begin{cases}\mu_{\ell-2}=\mu_{\ell-2}^*+\delta_2\\ \mu_{\ell-1}=\mu_{\ell-1}^*+\delta_1\\ 0\le\delta_2<\delta_1\le 2\delta_2\le 2^{\ell-2}\varepsilon\end{cases}\qquad\Longrightarrow\begin{cases}\mu_\ell=\mu_\ell^*+\delta\\ \delta_1<\delta\le 2\delta_1\le 2^{\ell-1}\varepsilon\end{cases}$$

we show that hp' implies also that $\mu_{\ell+1}>\mu_{\ell+1}^*$ and $\mu_{\ell+1}-\mu_\ell>\mu_{\ell+1}^*-\mu_\ell^*$ and $\mu_{\ell+1}\le 2\mu_\ell-\mu_{\ell-1}$. Note that, differently from the case of hp, here we have both an upper and lower bound on the next element of the sequence. This is because we have already shown from before that, if $a(\mu)$ is continuous at μ_ℓ , then $\mu_{\ell+1}\le 2\mu_\ell-\mu_{\ell-1}$. At $\ell=k$ this condition is satisfied by the assumption on ε. Furthermore, the initial conditions on hp' ensure that the condition will keep being satisfied at subsequent steps. Thus, we check the implication coming from hp' only for the case when μ_ℓ is not at a jump point. Since $\delta\le 2^{\ell-1}\varepsilon$, then $\mu_\ell<\mu_{\ell+1}^*$, and so $a(\mu_\ell)=A+\mu_\ell(S+\ell)D$, while the expectations are: $\beta(\mu_{\ell-1},\mu_\ell)=A+\frac{\mu_{\ell-1}+\mu_\ell}{2}(S+(\ell-1)D)+\frac{\mu_\ell^2-\mu_\ell^{*2}}{2(\mu_\ell-\mu_{\ell-1})}D$ and $\beta(\mu_\ell,\mu_{\ell+1})=A+\frac{\mu_{\ell+1}+\mu_\ell}{2}(S+\ell D)+\frac{\mu_{\ell+1}^2-\mu_{\ell+1}^{*2}}{2(\mu_{\ell+1}-\mu_\ell)}D$. We can check that the following inequality holds:

$$\beta(\mu_\ell,\mu_{\ell+1})<2a(\mu_\ell)-\beta(\mu_{\ell-1},\mu_\ell)\iff$$

$$(\mu_{\ell+1}+\mu_\ell)(S+\ell D)+\frac{\mu_\ell^2-\mu_\ell^{*2}}{\mu_\ell-\mu_{\ell-1}}D<(3\mu_{\ell-1}-\mu_{\ell-2})(S+\ell D)+\frac{\mu_{\ell-1}^{*2}-\mu_{\ell-2}^2}{\mu_{\ell-1}-\mu_{\ell-2}}D\iff$$

$$\frac{\mu_{\ell+1}^2-\mu_\ell^{*2}}{\mu_{\ell+1}-\mu_\ell}D<(\delta-\delta_1)(S+\ell D)+\frac{\mu_\ell^{*2}-\mu_{\ell-1}^2}{\mu_\ell-\mu_{\ell-1}}D$$

as $C\to 0$ we get: $(\delta-\delta_1)B>0$. Then, for local calibration, we must have that $\mu_{\ell+1}=\mu_{\ell+1}^*+\delta'$, for $\delta'>\delta$. However, recall also that local calibration requires $\delta'\le 2\delta$, then $\delta'^{\ell}\varepsilon$. Applying the argument by induction we show that $\mu_K>1$.

The argument works in a similar fashion for $B > 0$, keeping in mind that $a(\mu)$ will now be decreasing in μ. **Q.E.D.**

Example 2. The following three games are played each with probability $\frac{1}{3}$.

ω_1	L	M	R	ω_2	L	M	R	ω_3	L	M	R
U	$(1,1)$	$(0,0)$	$(1,0)$	U	$(.,0)$	$(1,1)$	$(.,0)$	U	$(0,1)$	$(2,0)$	$(0,0)$
D	$(0,0)$	$(2,0)$	$(0,1)$	D	$(.,0)$	$(0,1)$	$(.,0)$	D	$(1,0)$	$(0,0)$	$(1,1)$

with $K_1 = 2$ and $K_3 = 3$. For the clustering made by player 1, we identify L with 0, M with 1 and R with 3 on the real line and we identify a probability distribution $p_j \in \Delta(\{L, M, R\})$ with the resulting mean location on the line, i.e. $p_j(M) + 3p_j(R)$. The clustering is made using the Euclidean distance in this one-dimensional space.

There is a unique Nash equilibrium in each game (UL in ω_1; UM in ω_2; DR in ω_3) and it employs pure strategies.

If ω_1 and ω_2 are put together in one analogy class, then the resulting ABEE is unique and leads to DR in ω_1, thereby leading player 1 to cluster ω_1 with ω_3 (where R would be played).

If ω_2 and ω_3 are put together in one analogy class, then the resulting ABEE is unique and leads to UL in ω_3, thereby leading player 1 to cluster ω_3 with ω_1 (where L would be played).

If ω_1 and ω_3 are put together in one analogy class, then the ABEE is the same as the Nash equilibrium leading to an average value of $\frac{0+3}{2} = 1.5$ in this cluster, but then game ω_1 would have to be clustered with ω_2 given that 0 is closer to 1 than to 1.5. Thus there is no calibrated ABEE in this case.

Strategic Substitutes Discrete version

We illustrate how a distributional calibrated ABEE with only "interval" analogy partition would look like in the strategic substitutes environment when there are only three values of μ and players can use two categories $K = 2$. That is, we consider now μ to be a discrete random variable that takes value μ_s with probability $0 < p_s < 1$, where $s = 1, 2, 3$, $0 < \mu_1 < \mu_2 < \mu_3 < 1$ and $p_1 + p_2 + p_3 = 1$.

There are three possible ways to partition the state space into two analogy classes: $An^1 = \{\{\mu_1\}, \{\mu_2, \mu_3\}\}$ or $An^2 = \{\{\mu_2\}, \{\mu_1, \mu_3\}\}$ or $An^3 = \{\{\mu_3\}, \{\mu_1, \mu_2\}\}$. We think of the first and third analogy partitions as the discrete version of an interval partition, while the second partition cannot be related to intervals since the extreme values μ_1 and μ_3 are bundled together separately from the middle value μ_2. We refer to the probability distribution of agent i over analogy partitions as λ_i and we maintain the symmetry assumption on analogy partitions, so that $\lambda_1 = \lambda_2 \equiv \lambda$. We have:[31]

Proposition 9. *In the discrete version of the strategic substitutes environment there always exists a symmetric globally calibrated distributional ABEE. If $B \neq -AC$ and $\mu_2 \neq \frac{p_1\mu_1 + p_3\mu_3}{p_1+p_3}$, any globally calibrated distributional ABEE is such that only interval analogy partitions are in the support of λ ($An^2 \notin supp\lambda$).*

Proof. Let $\lambda^s \equiv \lambda(An^s)$, where $An^s = \{\{\mu_s\}, \{\mu_{s'}, \mu_{s''}\}\}$, for $s \neq s' \neq s'' \in \{1, 2, 3\}$. We show that there exists a (distributional) globally calibrated (GC) ABEE such that it assigns 0 probability to An^2, i.e. $\lambda^2 = 0$ (and that any such equilibrium must be such that $\lambda^2 = 0$).

We first show that any GC ABEE is such that $\lambda^2 = 0$.

First, consider the case with $\lambda^2 = 1$. The equilibrium strategies are the same as those identified for the continuous environment in Proposition 4. The local calibration requirements -which are necessary condition for GC- on μ_1 and μ_3 are satisfied iff $\mu_2 = \frac{p_1\mu_1+p_3\mu_3}{p_1+p_3}$, case that is excluded in our proposition.

Let $|supp[\lambda]| > 1$. Denote the total variance in analogy partition An by $Var_\lambda(An)$. That is, $Var_\lambda(An) = \sum_{\alpha \in An} p(\alpha) \sum p(\mu|\alpha)(\bar{a}(\mu) - \beta(\alpha))^2$. Given An^s, $\beta(\mu_s) = \bar{a}(\mu_s)$ and $\beta(\mu_{s'}, \mu_{s''}) = \frac{p_{s'}\bar{a}(\mu_{s'})+p_{s''}\bar{a}(\mu_{s''})}{p_{s'}+p_{s''}}$. Thus, the total variation in An^s can be expressed simply by $Var_\lambda(An^s) = \frac{p_{s'}p_{s''}}{p_{s'}+p_{s''}}(\bar{a}(\mu_{s'}) - \bar{a}(\mu_{s''}))^2$.

Note that the aggregate behavior, defined as in (2), must differ for different μ_s: if $\bar{a}(\mu_s) = \bar{a}(\mu_{s'}) = a$, then $\beta(\mu_s, \mu_{s'}) = a$ by consistency. Note that $|supp[\lambda]| > 1$ implies $\lambda^s > 0$ or $\lambda^{s'} > 0$. Take $\lambda^{s'} > 0$. Then, to satisfy local calibration, it must be the case that $\beta(\mu_s, \mu_{s''}) = a$ (otherwise μ_s would be reassigned). But then $\bar{a}(\mu_{s''}) = a$ and so $a(\mu|An^{\tilde{s}}) = A + \mu(B + Ca)$, for $\tilde{s} = 1, 2, 3$, and, in aggregate,

[31] We leave for future research whether in the substitute case, one can always (i.e., when there are more than three games and for general K values) find a calibrated distributional ABEE with support of analogy partitions contained in the subset of interval partitions.

$a = A + \mu(B + Ca)$. Hence, $\bar{a}(\mu_s) = \bar{a}(\mu_{s'})$ if and only if $\mu_s = \mu_{s'}$. Therefore, we can assume $\bar{a}(\mu_s) < \bar{a}(\mu_{s'}) < \bar{a}(\mu_{s''})$.

Let λ have full support. To show that λ is not GC wrt to $\bar{a}$, it suffices to show that $Var_\lambda(An^{s''}) < Var_\lambda(An^{s'})$. Since $\lambda^{s'} > 0$, $An^{s'}$ must be locally calibrated, which implies that $\bar{a}(\mu_{s'}) = \beta(\mu_s, \mu_{s''})$. Then, $Var_\lambda(An^{s'}) = p_s(\bar{a}(\mu_s) - \bar{a}(\mu_{s'}))^2 + p_{s''}(\bar{a}(\mu_{s''}) - \bar{a}(\mu_{s'}))^2$, whereas $Var_\lambda(An^{s''}) = \frac{p_s p_{s'}}{p_s + p_{s'}}(\bar{a}(\mu_s) - \bar{a}(\mu_{s'}))^2$. Since $p_s > \frac{p_s p_{s'}}{p_s + p_{s'}}$, then $Var_\lambda(An^{s'}) > Var_\lambda(An^{s''})$. Thus, λ cannot have full support.

Consider now λ with support of size 2 and $\lambda^2 > 0$. If $\bar{a}(\mu_2)$ is strictly between $\bar{a}(\mu_1)$ and $\bar{a}(\mu_3)$, the argument used in the full support case can be applied here too. We show that whenever $\bar{a}(\mu_2)$ is not between $\bar{a}(\mu_1)$ and $\bar{a}(\mu_3)$, we reach a contradiction. We focus on two cases, all other cases can be dealt with in a similar fashion: let $\lambda^3 = 0$ and either (i) $\bar{a}(\mu_2) < \bar{a}(\mu_1) < \bar{a}(\mu_3)$ or (ii) $\bar{a}(\mu_1) < \bar{a}(\mu_3) < \bar{a}(\mu_2)$. Since $\mu_s < 0$, for all s, the above inequalities imply that $a(\mu_1|An^1) > a(\mu_1|An^2)$, $a(\mu_2|An^1) < a(\mu_2|An^2)$ and $a(\mu_3|An^1) > a(\mu_3|An^2)$ in both cases (i) and (ii). If $B + C\beta(\mu_2, \mu_3) \geq 0$, in case (i) $a(\mu_1|An^1) \leq a(\mu_2|An^1)$ and so $\bar{a}(\mu_1) < \bar{a}(\mu_2)$ -contradiction- and in case (ii) $a(\mu_3|An^2) \geq a(\mu_2|An^2)$ and again a contradiction: $\bar{a}(\mu_3) > \bar{a}(\mu_2)$. Similarly, if $B + C\beta(\mu_2, \mu_3) < 0$, one can show that $\bar{a}(\mu_3) < \bar{a}(\mu_2)$ in case (i) and $\bar{a}(\mu_2) < \bar{a}(\mu_1)$ in case (ii). This concludes the argument for support of size 2.

Therefore, there is no globally calibrated ABEE such that $\lambda^2 > 0$.

We now show that there always exists a globally calibrated distributional ABEE with $\lambda = (\lambda^1, 0, 1 - \lambda^1)$. Note that $\bar{a}(\mu)$ is a continuous function of λ_1 for all μ. This can be checked through standard computations. Abusing notation, we write $Var_{\lambda^1}(An) \equiv Var_\lambda(An)$ and $Var_{\lambda^1}(An^s) = \frac{p_{s'} p_{s''}}{p_{s'} + p_{s''}}(\bar{a}(\mu_{s'}) - \bar{a}(\mu_{s''}))^2$ is also continuous in λ^1.

Note that if $\lambda^1 = 1$ and $Var_1(An^1) < Var_1(An^3)$, then $(a(\cdot|An^1), An^1)$ is a globally calibrated ABEE. Also, if $\lambda^1 = 0$ and $Var_0(An^3) < Var_0(An^1)$, then $(a(\cdot|An^3), An^3)$ is a globally calibrated ABEE. There might be values of μ_1, μ_2 and μ_3 such that neither analogy partition is GC. That is, $Var_1(An^1) - Var_1(An^3) > 0$ and $Var_0(An^1) - Var_0(An^3) < 0$. By continuity, there exists $\hat{\lambda}^1$ such that $Var_{\hat{\lambda}^1}(An^1) = Var_{\hat{\lambda}^1}(An^3)$. Note that this equality implies that $\bar{a}(\mu_2)$ is between $\bar{a}(\mu_1)$ and $\bar{a}(\mu_3)$, so $Var_{\hat{\lambda}^1}(An^1) < Var_{\hat{\lambda}^1}(An^2)$. Then, $(a, \hat{\lambda})$ is a globally calibrated distributional ABEE. **Q.E.D**